Polyharmonic and Related Kernels on Manifolds: Interpolation and Approximation [*†]

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Abstract

This article is devoted to developing a theory for effective kernel interpolation and approximation in a general setting. For a wide class of compact, connected C^∞ Riemannian manifolds, including the important cases of spheres and $SO(3)$, we establish, using techniques involving differential geometry and Lie groups, that the kernels obtained as fundamental solutions of certain partial differential operators generate Lagrange functions that are uniformly bounded and decay away from their center at an algebraic rate, and in certain cases, an exponential rate. An immediate corollary is that the corresponding Lebesgue constants for interpolation as well as for L_2 minimization are uniformly bounded with a constant whose only dependence on the set of data sites is reflected in the *mesh ratio*, which measures the uniformity of the data. The kernels considered here include the restricted surface splines on spheres, as well as surface splines for $SO(3)$, both of which have elementary closed-form representations that are computationally implementable. In addition to obtaining bounded Lebesgue constants in this setting, we also establish a "zeros lemma" for domains on compact Riemannian manifolds – one that holds in as much generality as the corresponding Euclidean zeros lemma (on Lipschitz domains satisfying interior cone conditions) with constants that clearly demonstrate the influence of the geometry of the boundary (via cone parameters) as well as that of the Riemannian metric.

1 Introduction

Radial basis functions (RBFs) have proven to be a powerful tool for analyzing scattered data on R^n. More recently, spherical basis functions (SBFs), which are analogs of RBFs on the n-sphere, and periodic basis functions (PBFs), which are analogs of RBFs on the n-torus, have had comparable success for analyzing scattered data on these manifolds. A theoretical

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drawback is that most RBFs are globally defined, thin plates splines are an example, and even those that are locally defined such as Wendland functions behave globally when approximating at densely packed data sites. Nevertheless, certain RBF approximants, in their numerical implementation, exhibit localized behavior, i.e., changing data locally only significantly alters the interpolant locally. Since the pioneering work of Duchon [9, 8], it has been a mystery why RBF/SBF approximants display local behavior even though the bases are globally supported. It was long suspected that there were "local bases" hidden within the space of translates of RBFs/SBFs.

A major objective of this paper is to establish that, for spheres and $SO(3)$, there are closed-form kernels whose associated approximation spaces possess highly localized bases, in the form of Lagrange functions for given scattered data. Our previous work [16] established the existence of such bases on compact manifolds, but the kernels we constructed did not have closed form.

We will carry out the construction and proofs that establish the existence and properties of these closed-form kernels in the context of of more general manifolds, thus achieving another objective: obtaining results for a broader class of kernels on manifolds than the ones we treated in [16], where only reproducing kernels for Sobolev spaces were dealt with.

We also address similar issues for conditionally positive kernels on manifolds, where a given space of functions is to be reproduced. In the case of $\mathbb{R}^d$, this involves little more than polynomial reproduction. For manifolds – even for spheres and $SO(3)$ – the kernels and spaces are not so simple, and new techniques are required to deal with the problem.

Goals Given a manifold $\mathbb{M}$, a finite set of points $X = \{x_1, \ldots, x_N\} \subset \mathbb{M}$ and a kernel $k\colon \mathbb{M} \times \mathbb{M} \to \mathbb{R}$, one may attempt to fit a smooth function using functions from $V_X := \operatorname{span}\{k(\cdot, x_j),\ x_j \in X\}$, or more generally, to use functions of the form

$$s = \sum_{j=1}^{N} a_j k(\cdot, x_j) + p, \tag{1.1}$$

where the supplementary function p comes from a simple space (like polynomials or spherical harmonics). The framework described above applies to fitting data by means of interpolation, least squares or near interpolation with a smoothing term.

This article is devoted to developing a theory for effective kernel approximation in a general setting. The problem is described as follows: we seek kernels $k\colon \mathbb{M}\times\mathbb{M} \to \mathbb{R}$ for which interpolation is well posed and that have a convenient closed form representation allowing for effective computations. Furthermore, we are interested in aspects of the interpolants/approximants concerning stability, locality and so on.

In [16] and [15], we developed a theory for compact Riemannian manifolds using positive definite "Sobolev kernels." The theory developed there addresses and answers questions concerning properties of bases for V_X, properties related to locality, stability of approximation and interpolation, and other matters. In this paper, these questions, which are listed below, are addressed and answered for a broad class of kernels on $\mathbb{M}$ that are Green's functions for certain elliptic operators, and, when the manifold is a sphere, real projective space or $SO(3)$, are computationally implementable as well.

Locality. Are there local bases for V_X? That is, are there bases similar to those for wavelet systems or B-splines [6, Chapter 5]? At a minimum, we would like a basis $\{v_j\}$ to satisfy $|v_j(x)| \le r(d(x, x_j))$, with r a rapidly decaying function.

L_p **conditioning**. Are there bases that are well conditioned in L_p, after renormalization? That is, can we find bases for which there constants c_1, c_2 such that $c_1\|a\|_{\ell_p} \le \|\sum_{j=1}^N a_j v_j\|_{L_p} \le c_2\|a\|_{\ell_p}$, with c_1, c_2 independent of N, and, after a suitable normalization, independent of p?

Marcinkiewicz-Zygmund property. Does the space V_X possess a Marcinkiewicz-Zygmund property relating samples to the size of the function? For $s \in V_X$, this means that the norms $\|j \mapsto s(x_j)\|_{\ell_p}$ and $\|s\|_{L_p}$ are equivalent, with constants involved independent of N.

Stability of interpolation. Is interpolation stable? Is the Lebesgue constant bounded or, more generally, is the p norm of the interpolant controlled by the ℓ_p norm of the data?

Stability of approximation in L_p. Is approximation by L_2 projection stable in L_p? Here, $1 \le p \le \infty$. In particular, what we want is that the orthogonal projector with range V_X be continuously extended to each L_p, and that it has bounded operator norm independent of N.

The Sobolev kernels we dealt with in [16] do not possess simple, closed form representations, even when the underlying manifold is a sphere; they are defined indirectly, as reproducing kernels for certain Sobolev spaces. To applied effectively to data fitting problems, such a kernel should have an implementable characterization, by which interpolation, approximation or other computational problems can be treated. In the important cases relating to spheres and $SO(3)$, we exhibit computationally implementable kernels. In particular, these kernels include restricted surface splines on spheres, and surface splines on $SO(3)$, both of which have simple closed form representations. Furthermore, for both of these cases, theoretical approximation results concerning direct theorems, inverse theorems, and Bernstein inequalities are known to hold [22]. In conjunction with the stability of interpolation and least-squares approximation in L_p, both yield new, precise error estimates for these implementable schemes.

Kernels The class of kernels considered in this paper are those kernels κ that act as fundamental solutions for elliptic differential operators of the form $\mathcal{L}_m = \prod_j(\Delta - r_j)$ and lower order perturbations of these. The origin of this approach lies in the work of Duchon, [9, 8] on surface splines, where the underlying kernel is the Green's functions for iterated Laplacian, Δ^m, on $\mathbb{R}^d$. Such kernels have also been used on Riemannian manifolds, [10, 27]. For this reason, we call them *polyharmonic*; see Definition 3.2. Throughout this article, we use k_m to denote a generic polyharmonic kernel.

This is a classic family of kernels and is sufficiently robust to include many interesting examples. For instance, such kernels have also been in use for some time on spheres, and have formed one of the earliest families of SBFs (see [11] and references for a list of early examples). In this setting, certain careful choices of these kernels result in the complete family of surface splines restricted to the sphere[1], introduced in [24], which we define below in (3.2) and denote in the "zonal" form as $k_m(x, t) = \phi_s(x \cdot t)$. Here s is related to m via $m = s + d/2$.

[1]A related problem, which can be considered a generalization of this particular set up has recently been

It also includes the surface splines on $SO(3)$, introduced in [17] and defined below in (3.4), and denoted throughout the paper by $\mathbf{k}_m$.

A second type of kernel, ostensibly different from the polyharmonic kernels, are the Sobolev (or Matérn) kernels, which have been introduced for compact Riemannian manifolds in [16]. These kernels come about as reproducing kernels for Sobolev spaces. We denote such kernels by $\kappa_m(x, y)$, where m indicates the order of the Sobolev space. A corollary of the results presented in this paper is that, in many cases, the Sobolev kernels are in fact polyharmonic kernels. There is an operator $\mathcal{L}_m$ for which $\kappa_m(x, y)$ is the fundamental solution.

Kernel	**Notation**	**Location in manuscript**
Restricted surface spline	$(x, y) \mapsto \phi_s(x \cdot y)$	Example 2 in 3.2
Surface spline on SO(3)	$\mathbf{k}_m$	Example 3 in 3.2
Polyharmonic kernel	k_m	Definition 3.2
Sobolev kernel	κ_m	[16, 3.3]

Table 1: Index for kernels used

Outline The layout of this paper is as follows. Following the introduction and background, Section 2 deals with certain geometric notions relevant to this article. Section 3 treats interpolation by *conditionally positive definite* kernels, the function spaces that they generate, and the nature of their interpolants. We discuss some important examples on well known manifolds, including spheres and the rotation group. Finally we define precisely the polyharmonic kernels, which are the kernels that we treat in our main results; they include the examples we provided earlier. We demonstrate that they are conditionally positive definite, identify the seminorm of the native spaces associated with these kernels, and discuss the variational problem associated with their interpolants.

The relationship between the polyharmonic kernels and the Sobolev kernels of [16] will be covered in Section 4. We show that, under certain conditions, the polyharmonic operators $\mathcal{L}_m$ can be expressed as combinations of operators generated by covariant derivatives, and vice versa (this is done in Lemma 4.3). This allows us to conclude that Sobolev kernels are examples of polyharmonic kernels. On the other hand, it permits us to demonstrate, in Section 4.2, that the native space seminorms associated with polyharmonic kernels exhibit the same behavior (metric equivalence to Euclidean Sobolev seminorms, zeros lemmas, etc.) as the native space norms associated with Sobolev kernels.

The main results of the paper are given in Section 5. Namely, the Lagrange function associated with a kernel k_m is rapidly decaying, and gives rise to a uniformly bounded Lebesgue constant and a uniformly bounded L_2 minimization projector. The properties mentioned

considered by Fuselier and Wright, [12]. There, kernel interpolation is considered on manifolds that are embedded in $\mathbb{R}^d$ by using the restriction of various other RBFs to the manifold – this is accomplished by constructing interpolants in the ambient Euclidean space and then restricting these to the manifold. (In contrast, we work directly with the manifold, making use of its intrinsic structure.)

above – concerning locality, stability, conditioning, and so on – then follow immediately. We then discuss implications of this for surface spline kernels on spheres and $SO(3)$.

Essential to our proofs in Section 5 are theorems giving L_p Sobolev-space estimates for functions having zeros quasi-uniformly distributed on a domain Ω, with $\partial\Omega$ being Lipschitz. Such theorems may hold both in $\mathbb{R}^d$ and on $\mathbb{M}$ itself, and in Section A, we treat both cases. For the case of a manifold $\mathbb{M}$, these theorems involve geometric ideas; in particular, they require use of a *minimal ε-set* of points in $\mathbb{M}$ (cf. [13]), which replaces a simpler set in $\mathbb{R}^d$. The results for $\mathbb{M}$ turn out to be intrinsic and hold in the same generality as those in the Euclidean case. The bounds and the condition on the meshnorm reflect geometric properties – particularly, parameters from the cone condition on $\partial\Omega$, properties of the manifold $\mathbb{M}$, and parameters of the Sobolev spaces themselves – but are independent of the volume and diameter of Ω.

2 Background

We now discuss some relevant details about analysis on compact, complete, connected C^∞ Riemannian manifolds. This is the same setting as [16]. Refer to it for a more detailed treatment and further references.

Throughout our discussion, we will assume that $(\mathbb{M}, g)$ is a d-dimensional, connected, complete C^∞ Riemannian manifold without boundary; the Riemannian metric for $\mathbb{M}$ is g, which defines an inner product $g_p(\cdot,\cdot) = \langle\cdot,\cdot\rangle_{g,p}$ on each tangent space T_pM; the corresponding norm is $|\cdot|_{g,p}$. As usual, a chart is a pair $(\mathfrak{U}, \phi)$ such that $\mathfrak{U} \subset \mathbb{M}$ is open and the map $\phi : \mathfrak{U} \to \mathbb{R}^d$ is a one-to-one homeomorphism. An atlas is a collection of charts $\{(\mathfrak{U}_\alpha, \phi_\alpha)\}$ indexed by α such that $\mathbb{M} = \cup_\alpha \mathfrak{U}_\alpha$ and, when $\phi_\alpha(\mathfrak{U}_\alpha) \cap \phi_\beta(\mathfrak{U}_\beta) \neq \emptyset$, $\phi_\beta \circ \phi_\alpha^{-1}$ is C^∞. In a fixed chart $(\mathfrak{U}, \phi)$, the points $p \in \mathfrak{U}$ are parametrized by $p = \phi^{-1}(x)$, where $x = (x^1, \ldots, x^d) \in U = \phi(\mathfrak{U})$.

In these local coordinates, TM_p and T^*M_p, the tangent and cotangent spaces at p, have bases comprising the vectors $\mathbf{e}_j = \left(\frac{\partial}{\partial x^j}\right)_p$, $j = 1 \ldots d$ and $\mathbf{e}^k = (dx^k)_p$, $k = 1 \ldots d$, respectively. These vary smoothly over $U = \phi(\mathfrak{U})$ and form dual bases in the sense that $\mathbf{e}^k(\mathbf{e}_j) = \frac{\partial x^k}{\partial x^j} = \delta^k_j$. In the usual way, the inner product $\langle\cdot,\cdot\rangle_{g,p}$ provides an isomorphism between the cotangent and tangent spaces. Thus, regarding $\mathbf{e}^k$'s as vectors, we have that $\langle \mathbf{e}^k, \mathbf{e}_j\rangle_{g,p} = \delta^k_j$. A vector $\mathbf{v}$ can be represented either as $\mathbf{v} = \sum_j v^j \mathbf{e}_j$ or as $\mathbf{v} = \sum_k v_k \mathbf{e}^k$; the v^j's and v_k's are the *contravariant* and *covariant* components of $\mathbf{v}$, respectively. Relative to these bases, the inner product $\langle\cdot,\cdot\rangle_{g,p}$ has the form

$$\langle \mathbf{u}, \mathbf{v}\rangle_{g,p} = \sum_{i,j=1}^{d} g_{ij} u^i v^j = \sum_{i,j=1}^{d} g^{ij} u_i v_j\ , \text{ where } g_{ij} = \langle \mathbf{e}_i, \mathbf{e}_j\rangle_{g,p} \text{ and } g^{ij} = \langle \mathbf{e}^i, \mathbf{e}^j\rangle_{g,p} \tag{2.1}$$

The matrices (g_{ij}) and (g^{ij}) are inverse to each other, and are of course symmetric and positive definite. The inner product $\langle \mathbf{v}, \mathbf{w}\rangle_{g,p}$ is itself independent of coordinates. In addition, if $\mathbf{v}$ and $\mathbf{w}$ are C^∞ vectors fields in p, then $\langle \mathbf{v}, \mathbf{w}\rangle_{g,p}$ is also C^∞.

The metric g also induces an invariant volume measure $d\mu$ on $\mathbb{M}$. The local form of the measure is $d\mu(x) = \sqrt{\det(g)}dx^1 \cdots dx^d$, where $\det(g) = \det(g_{ij})$.

Geodesics are curves $\gamma : \mathbb{R} \to \mathbb{M}$ that locally minimize the arc length functional, $\int_a^b |\dot\gamma|_{g,p} dt$. If we use the arc length s as the parameter (i.e., $t \to s$), then, in local coordinates, a geodesic

satisfies the Euler-Lagrange equations:

$$\frac{d^2x^k}{ds^2} + \sum_{i,j=1}^{d} \Gamma^k_{ij} \frac{dx^i}{ds}\frac{dx^j}{ds} = 0, \quad \text{where } \Gamma^k_{ij} := \frac{1}{2} \sum_{m\in\{1\dots d\}} g^{km} \left(\frac{\partial g_{jm}}{\partial x^i} + \frac{\partial g_{im}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^m} \right). \quad (2.2)$$

The Γ^k_{ij} are the *Christoffel symbols.*

A geodesic solving (2.2) is specified by giving an initial point $p \in \mathbb{M}$, whose coordinates we may take to be $\big(x^1(0), \dots, x^d(0)\big) = 0$, together with a tangent vector $\mathbf{t}_p$ having components $\frac{dx^i}{ds}(0)$. A Riemannian manifold is said to be *complete* if the geodesics are defined for all values of the parameter s. All compact Riemannian manifolds without boundary are complete, and so are many non-compact ones, including $\mathbb{R}^d$.

We define the *exponential map* $\mathrm{Exp}_p : T_pM \to \mathbb{M}$ by letting $\mathrm{Exp}_p(0) = p$ and $\mathrm{Exp}_p(s\mathbf{t}_p) = \gamma_p(s)$, where $\gamma_p(s)$ is the unique geodesic that passes through p for $s = 0$ and has a tangent vector $\dot\gamma_p(0) = \mathbf{t}_p$ of length 1; i.e., $\langle \mathbf{t}_p, \mathbf{t}_p\rangle_{gp}. = 1$.

Although geodesics having different initial, non-parallel unit tangent vectors $\mathbf{t}_p = \dot\gamma_p(0)$ may eventually intersect, there will always be a neighborhood $\mathfrak{U}_p$ of p where they do not. In $\mathfrak{U}_p$, the initial direction $\mathbf{t}_p$, together with the arc length s, uniquely specify a point q via $q = \gamma_p(s)$, and the exponential map Exp_p is a diffeomorphism between the corresponding neighborhoods of 0 in T_pM and p in M. In particular, there will be a largest ball $B(0, \mathrm{r}_p) \in T_pM$ about the origin in T_pM such that $\mathrm{Exp}_p : B(0, \mathrm{r}_p) \to \mathbf{b}(p, \mathrm{r}_p) \subset \mathbb{M}$ is injective and thus a diffeomorphism; r_p is called the *injectivity radius for p*. By choosing cartesian coordinates on $B(0, \mathrm{r}_p)$, with origin 0, and using the exponential map, we can parametrize $\mathbb{M}$ in a neighborhood of p via $q = \mathrm{Exp}_p(x)$, $x \in T_pM$.

The *injectivity radius* of $\mathbb{M}$ is $\mathrm{r}_{\mathbb{M}} := \inf_{p\in\mathbb{M}} \mathrm{r}_p$. If $0 < \mathrm{r}_{\mathbb{M}} \le \infty$, the manifold is said to have *positive* injectivity radius. For any $r < \mathrm{r}_{\mathbb{M}}$ and any $p \in \mathbb{M}$, the exponential map $\mathrm{Exp}_p : B(0, r) \to \mathbf{b}(p, r)$ is a diffeomorphism.

We make special note of the fact that, for a compact Riemannian manifold, the family of exponential maps are uniformly isomorphic; i.e., there are constants $0 < \Gamma_1 \le 1 \le \Gamma_2 < \infty$ so that for every $p_0 \in \mathbb{M}$ and every $x, y \in B(0, \mathrm{r})$, where $\mathrm{r} \le \mathrm{r}_{\mathbb{M}}/3$,

$$\Gamma_1|x - y| \le \mathrm{dist}(\mathrm{Exp}_{p_0}(x), \mathrm{Exp}_{p_0}(y)) \le \Gamma_2|x - y|. \quad (2.3)$$

When we use Γ_1 and Γ_2 in this paper, we always assume that there is some fixed radius smaller that $\mathrm{r}_{\mathbb{M}}$ on which they are computed. This avoids a tiresome repetition of this fact throughout the paper.

An order k *covariant tensor* $\mathbf{T}$ is a real-valued, multilinear function of the k-fold tensor product of T_pM. We denote by T^k_pM the covariant tensors of of order k at p. In terms of the local coordinates, there is a smoothly varying basis $\mathbf{e}^{i_1} \otimes \cdots \otimes \mathbf{e}^{i_k}$ for the k-fold tensor product of tangent spaces. Thus, the covariant tensor field $\mathbf{T}$ of order k on $\mathfrak{U}$ can be written as

$$\mathbf{T} = \sum_{\hat\imath\in\{1\dots d\}^k} T_{\hat\imath}\, \mathbf{e}^{i_1} \otimes \cdots \otimes \mathbf{e}^{i_k},$$

where we adopt the convention $\hat{\imath} = (i_1, \ldots, i_k)$. The $T_{\hat{\imath}}$ are the covariant components of $\mathbf{T}$. The metric g_{ij} is itself an order 2 covariant tensor field. One can also define contravariant tensors and tensors of mixed type.

Because $T_p^k M = T_p M \otimes \cdots \otimes T_p M$ (k times), the metric g induces a natural, useful, invariant inner product on $T_p^k M$; in terms of covariant components, it is given by

$$\langle \mathbf{S}, \mathbf{T} \rangle_{g,p} = \sum_{\hat{\imath},\hat{\jmath} \in \{1\ldots d\}^k} g^{i_1 j_1} \cdots g^{i_k j_k} S_{\hat{\imath}} T_{\hat{\jmath}} \,. \tag{2.4}$$

The *covariant derivative*, or *connection*, ∇ associated with $(\mathbb{M}, g)$ is defined as follows. If $\mathbf{T}$ is an order k (covariant) tensor, then the covariant derivative of $\mathbf{T}$ is

$$\nabla \mathbf{T} = \sum_{j \in \{1\ldots d\}} \sum_{\hat{\imath} \in \{1\ldots d\}^k} \underbrace{\left(\frac{\partial T_{\hat{\imath}}}{\partial x^j} - \sum_{r=1}^{k} \sum_{s \in \{1\ldots d\}} \Gamma^s_{j,i_r} T_{i_1,\ldots,i_{r-1},s,i_{r+1},\ldots,i_k} \right)}_{(\nabla \mathbf{T})_{\hat{\imath},j}} \mathbf{e}^{i_1} \otimes \cdots \otimes \mathbf{e}^{i_k} \otimes \mathbf{e}^{j},$$

which is an order $k+1$ covariant tensor with components $(\nabla \mathbf{T})_{\hat{\imath},j}$. The Γ^k_{ij} are the Christoffel symbols defined earlier.

A smooth function $f : \mathbb{M} \to \mathbb{R}$ is a 0 order tensor and so ∇f, which is the gradient of f is an order 1 tensor, $\nabla^2 f$ is an order 2 tensor. Continuing in this way, we may form $\nabla^k f$, which is an invariant version of the ordinary k^{th} gradient of a function on $\mathbb{R}^d$. (The superscript k here is an operator power, not a contravariant index.) The components of the k^{th} covariant derivative of f have the form

$$(\nabla^k f(x))_{\hat{\imath}} = (\partial^k f(x))_{\hat{\imath}} + \sum_{m=1}^{k-1} \sum_{\hat{\jmath} \in \{1\ldots d\}^m} A^{\hat{\jmath}}_{\hat{\imath}}(x) (\partial^m f(x))_{\hat{\jmath}} \tag{2.5}$$

where

$$(\partial^m f)_{\hat{\jmath}} := \frac{\partial^m}{\partial x^{j_1} \cdots \partial x^{j_m}} f \circ \phi^{-1},$$

and where the coefficients $x \mapsto A^{\hat{\jmath}}_{\hat{\imath}}(x)$ depend on the Christoffel symbols and their derivatives to order $k-1$, and, hence, are smooth in x. This can also be written in standard multi-index notation. Let $\alpha_1, \alpha_2, \ldots, \alpha_d$ be the number of repetitions of $1, 2, \ldots, d$ in $\hat{\jmath}$, and let $\alpha = (\alpha_1, \ldots, \alpha_d)$. Then,

$$(\partial^m f)_{\hat{\jmath}} := \frac{\partial^m}{\partial (x^1)^{\alpha_1} \cdots \partial (x^d)^{\alpha_d}} f \circ \phi^{-1} =: D^{|\alpha|}_{\alpha} f \circ \phi^{-1}, \ |\alpha| = \sum_{k=1}^{d} \alpha_k = m. \tag{2.6}$$

Another important quantity that we need to deal with is the *adjoint* of the covariant derivative ∇^*. This operator is defined by $\int_{\mathbb{M}} \langle \nabla^* \mathbf{T}, \mathbf{S} \rangle_{g,p} d\mu = \int_{\mathbb{M}} \langle \mathbf{T}, \nabla \mathbf{S} \rangle_{g,p} d\mu$, where the inner product is given by (2.4), and it takes an order $k+1$ tensor to an order k tensor. The coordinate form of $\nabla^* \mathbf{T}$ is obtained via integration by parts:

$$(\nabla^* \mathbf{T})_{\hat{\imath}} = - \sum_{i,j} g^{jk} (\nabla \mathbf{T})_{\hat{\imath},j,k} \tag{2.7}$$

We can combine covariant derivatives and their adjoints to get scalar operators. In particular, if $f : \mathbb{M} \to \mathbb{R}$ is C^∞, then $\nabla^k f$ is an order k tensor. By applying ∇^* to it, we get $(\nabla^*)^k \nabla^k f$, which is a scalar. (Note that $(\nabla^*)^k = (\nabla^k)^*$.) The *Laplace–Beltrami operator* $\Delta := -\nabla^*\nabla$. In coordinates, again letting $\det(g) = \det(g_{ij})$, we have that

$$\Delta f = \det(g)^{-1/2} \sum_{i,j} \frac{\partial}{\partial x^i} \left(\det(g)^{1/2} g^{ij} \frac{\partial f}{\partial x^j} \right).$$

Sobolev spaces on subsets of $\mathbb{M}$ Sobolev spaces on subsets of a Riemannian manifold can be defined in an invariant way, using covariant derivatives [1]. In defining them, we will need to make use of the spaces L_p, L_q. To avoid problems with notation, we will use the sans-serif letters p, q, rather than p, q, as subscripts. Here is the definition.

Definition 2.1 ([1, p. 32])**.** *Let $\Omega \subset \mathbb{M}$ be a measurable subset. For all $1 \le \mathsf{p} \le \infty$, we define the Sobolev space $W_\mathsf{p}^m(\Omega)$ to be all $f : \mathbb{M} \to \mathbb{R}$ such that, for $0 \le k \le m$, $|\nabla^k f|_{g,p}$ in $L_\mathsf{p}(\Omega)$. The associated norms are as follows:*

$$\|f\|_{W_\mathsf{p}^m(\Omega)} := \begin{cases} \left(\sum_{k=0}^m \int_\Omega |\nabla^k f|_{g,p}^\mathsf{p} \, \mathrm{d}\mu(p)\right)^{1/\mathsf{p}}, & 1 \le \mathsf{p} < \infty; \\ \max_{0 \le k \le m} \left\| |\nabla^k f|_{g,p} \right\|_{L_\infty(\Omega)}, & \mathsf{p} = \infty. \end{cases} \tag{2.8}$$

When $\mathsf{p} = 2$, the norm comes from the Sobolev inner product

$$\langle f, g \rangle_{m,\Omega} := \langle f, g \rangle_{W_2^m(\Omega)} := \sum_{k=0}^m \int_\Omega \left\langle \nabla^k f, \nabla^k g \right\rangle_{g,p} \mathrm{d}\mu(p). \tag{2.9}$$

We also write the $\mathsf{p} = 2$ Sobolev norm as $\|f\|_{m,\Omega}^2 := \langle f, f \rangle_{m,\Omega}$. When $\Omega = \mathbb{M}$, we may suppress the domain: $\langle f, g \rangle_m = \langle f, g \rangle_{m,\mathbb{M}}$ and $\|f\|_m = \|f\|_{m,\mathbb{M}}$.

Metric equivalence The exponential map allows us to compare the Sobolev norms we've just introduced, to standard Euclidean Sobolev norms as follows:

Lemma 2.2 ([16, Lemma 3.2])**.** *For $m \in \mathbb{N}$ and $0 < r < \mathrm{r}_\mathbb{M}/3$, there are constants $0 < c_1 < c_2$ so that for any measurable $\Omega \subset B_r$, for all $j \in \mathbb{N}$, $j \le m$, and for any $p_0 \in \mathbb{M}$, the equivalence*

$$c_1 \|u \circ \mathrm{Exp}_{p_0}\|_{W_\mathsf{p}^j(\Omega)} \le \|u\|_{W_\mathsf{p}^j(\mathrm{Exp}_{p_0}(\Omega))} \le c_2 \|u \circ \mathrm{Exp}_{p_0}\|_{W_\mathsf{p}^j(\Omega)}$$

holds for all $u : \mathrm{Exp}_{p_0}(\Omega) \to \mathbb{R}$. The constants c_1 and c_2 depend on r, m and p, but they are independent *of Ω and p_0.*

Besov spaces on $\mathbb{M}$ Besov spaces can be defined and characterized in many equivalent ways. For a discussion, see Triebel's book [28, 1.11, and Chapter 7] and the references therein. Our definition follows Triebel's.

Definition 2.3. *For $0 < s \le m$ and $1 \le p < \infty$, we define the Besov space $B^s_{p,\infty}(\mathbb{M})$ as the collection of functions in $L_p(\mathbb{M})$ for which the following expression*

$$\|f\|_{B^s_{p,\infty}(\mathbb{M})} := \sup_{t>0} t^{-s} K(f,t)$$

is finite, where the K-functional $K(f,\cdot) : (0,\infty) \to (0,\infty)$ is defined as

$$K(f,t) := \inf \left\{ \|f-g\|_{L_p} + t^{2m} \|g\|_{W_p^m(\mathbb{M})} : g \in W_p^m(\mathbb{M}) \right\}.$$

For $p = \infty$, the definition is the same after substituting $L_p(\mathbb{M})$ by $C(\mathbb{M})$ and $W_p^m(\mathbb{M})$ by $C^m(\mathbb{M})$.

2.1 Notation

In order to distinguish balls on $\mathbb{R}^d$ from those in $\mathbb{M}$, we denote the ball centered at $p \in \mathbb{M}$ having radius r by $\mathbf{b}(p,r)$. (Euclidean balls are denoted $B(x,r)$.) Given a finite set $\Xi \subset \mathbb{M}$, we define its *mesh norm* (or *fill distance*) h and the *separation distance* q to be:

$$h := \sup_{p \in \mathbb{M}} \operatorname{dist}(p,\Xi) \qquad \text{and} \qquad q := \inf_{\substack{\xi,\zeta \in \Xi \\ \xi \neq \zeta}} \operatorname{dist}(\xi,\zeta). \tag{2.10}$$

The mesh norm measures the density of Ξ in $\mathbb{M}$, the separation radius determines the spacing of Ξ. The *mesh ratio* $\rho := h/q$ measures the uniformity of the distribution of Ξ in $\mathbb{M}$. If ρ is bounded, then we say that the point set Ξ is quasi-uniformly distributed, or simply that Ξ is quasi-uniform.

3 Interpolation by Kernels

The purpose of this section is to discuss further this interpolation problem and to present the kernels we employ. The kernels we consider are fundamental solutions of certain elliptic PDEs. They happen also to be conditionally positive definite, a well known class for which interpolation is understood. In particular, interpolation is well posed, and has a dual nature, as best interpolation from a function space.

In 3.1 we discuss interpolation with conditionally positive definite kernels, and present the associated problem of best interpolation. In 3.2 we present some motivating examples for our problem: surface spline interpolation on spheres and on $SO(3)$. In 3.3 we give the formal definition of the kernels we use and the operators they invert; we also discuss the associated variational problem they solve.

3.1 Interpolation with conditionally positive definite kernels

The kernels we consider in this article are conditionally positive definite on the compact Riemannian manifold. As a reference on this topic, we suggest [10, Section 4].

Definition 3.1. *A kernel is conditionally positive definite with respect to a finite dimensional space* Π *if, for any set of centers* Ξ*, the matrix* $\big(k(\xi,\zeta)\big)_{\zeta,\xi\in\Xi}$ *is positive definite on the subspace of all vectors* $\alpha\in\mathbb{C}^\Xi$ *satisfying* $\sum_{\xi\in\Xi}\alpha_\xi p(\xi)=0$ *for* $p\in\Pi$.

This is a very general definition which we will make concrete in the next subsections. Given a complete orthonormal basis $(\phi_j)_{j\in\mathbb{N}}$, of continuous functions, normalized in L_∞ (i.e., $\|\phi_j\|_\infty=1$) any kernel

$$k(x,y):=\sum_{j\in\mathbb{N}}\tilde{k}(j)\varphi_j(x)\overline{\varphi_j(y)}$$

with coefficients $\tilde{k}\in\ell_1(\mathbb{N})$ for which all but finitely many coefficients $\tilde{k}(j)$ are positive is conditionally positive definite with respect to $\Pi_{\mathcal{J}}=\mathrm{span}(\phi_j\mid j\in\mathcal{J})$, where $\mathcal{J}=\{j\mid\tilde{k}(j)\le 0\}$, since, evidently,

$$\begin{aligned}\sum_{\xi\in\Xi}\sum_{\zeta\in\Xi}\alpha_\xi k(\xi,\zeta)\overline{\alpha_\zeta} &= \sum_{\xi\in\Xi}\sum_{\zeta\in\Xi}\alpha_\xi\overline{\alpha_\zeta}\left(\sum_{j\in\mathbb{N}}\tilde{k}(j)\phi_j(\xi)\overline{\phi_j(\zeta)}\right)\\ &= \sum_{j\in\mathbb{N}}\tilde{k}(j)\sum_{\xi,\zeta\in\Xi}\alpha_\xi\phi_j(\xi)\overline{\alpha_\zeta\phi_j(\zeta)}=\sum_{j\notin\mathcal{J}}\tilde{k}(j)\|\alpha\phi_j\|^2_{\ell_2(\Xi)}>0\end{aligned}$$

provided $\sum_\xi\alpha_\xi\phi_j(\xi)=0$ for j satisfying $\tilde{k}(j)\le 0$.

In this case if the set of centers $\Xi\subset\mathbb{M}$ is unisolvent with respect to $\Pi_{\mathcal{J}}=\mathrm{span}(\varphi_j\mid j\in\mathcal{J})$ (meaning that $p\in\Pi_{\mathcal{J}}$ and $p(\xi)=0$ for $\xi\in\Xi$ implies that $p=0$) then the system of equations

$$\begin{cases}\sum_{\xi\in\Xi}a_\xi k(\zeta,\xi)+\sum_{j\in\mathcal{J}}b_j\varphi_j(\zeta)=y_\zeta & \zeta\in\Xi\\ \sum_{\xi\in\Xi}a_\xi\varphi_j(\xi)=0 & j\in\mathcal{J}\end{cases}$$

has a unique solution in $\mathbb{C}^\Xi\times\mathbb{C}^{\mathcal{J}}$ for each data sequence $\big(y_\zeta\big)_{\zeta\in\Xi}\in\mathbb{C}^\Xi$. When data is sampled from a continuous function at the points Ξ (i.e., $y_\zeta=f(\zeta)$) this solution generates a continuous interpolant:

$$I_\Xi f=I_{k,\mathcal{J},\Xi}f=\sum_{\xi\in\Xi}a_\xi k(\cdot,\xi)+\sum_{j\in\mathcal{J}}b_j\varphi_j$$

with the property that it is the minimizer of the semi-norm $|||\cdot|||_{k,\mathcal{J}}$, called the "native space" norm, given by

$$\left|\left|\left|\sum_{j\in\mathbb{N}}\hat{u}(j)\varphi_j\right|\right|\right|^2_{k,\mathcal{J}}=\sum_{j\notin\mathcal{J}}\frac{|\hat{u}(j)|^2}{\tilde{k}(j)},\tag{3.1}$$

over all functions $u=\sum_{j\in\mathbb{N}}\hat{u}(j)$ for which $u(\xi)=y_\xi$, $\xi\in\Xi$. If k is conditionally positive definite with respect to the set $\Pi_{\mathcal{J}}$, it will be conditionally positive definite with respect to $\Pi_{\mathcal{J}'}$ for any $\mathcal{J}'\supset\mathcal{J}$. For this reason, the interpolant and norm both are decorated by k and $\mathcal{J}$.

This has the consequence that any two conditionally positive definite kernels k,k' which have eigenfunction expansions that coincide on all but finitely many indices (say $\mathcal{I}$), produce the same interpolants. That is: $I_{k,\mathcal{I},\Xi}=I_{k',\mathcal{I},\Xi}$.

3.2 Examples of conditionally positive definite kernels

Example 1 (Surface Splines). As a first example of a conditionally positive definite kernel, we take $\mathbb{M} = \mathbb{R}^d$, and consider the kernels $k_m(x, \alpha) = \boldsymbol{\phi}_s(x - \alpha)$ given by the functions

$$\boldsymbol{\phi}_s(x) = C_{m,d} \begin{cases} |x|^{2s} \log |x| & d \text{ is even} \\ |x|^{2s} & d \text{ is odd} \end{cases}$$

where $s + d/2 = m$. For a certain $C_{m,d} \neq 0$, this is a fundamental solution for the operator Δ^m.

Because of the positivity of the generalized Fourier transform, one can see that $\boldsymbol{\phi}_s$ is conditionally positive definite on $\mathbb{R}^d$ with respect to Π_{m-1}. These have been studied by Duchon [9, 8], and they comprise some of the earliest and most popular examples of conditionally positive definite kernels.

Although our focus in this paper is on kernels on compact manifolds, the family of surface splines acts as a useful benchmark, since they have a simple, direct representation, as well as being conditionally positive definite, not to mention that for certain interpolation problems, their Lagrange functions decay rapidly (this was demonstrated in a least squares sense by Matveev in [21, Lemma 5] and pointwise in [16]) and have a uniformly bounded Lebesgue constant (cf. [16]).

Example 2 (Restricted Surface Splines on $\mathbb{S}^d$). When $\mathbb{M} = \mathbb{S}^d$, the eigenvalues of the Laplace–Beltrami operator are $\mu_\ell = \ell(\ell + d - 1)$ and each eigenvalue has $N(d, \ell) = \frac{(2d+\ell)\Gamma(\ell+d-1)}{\Gamma(\ell+1)\Gamma(d)}$ linearly independent eigenfunctions, the spherical harmonics $Y_{\ell,m}$.

We now introduce a family of kernels known as the restricted surface splines. These are kernels indexed by $m \in \mathbb{N}$, $m > d/2$. By writing $m = s + d/2$, we give the zonal expression

$$\phi_s(t) := \begin{cases} |1 - t|^s \log |1 - t| & s \in \mathbb{N} \\ |1 - t|^s & s \in \mathbb{N} + 1/2. \end{cases} \tag{3.2}$$

When d is even, s is integral and the first formula is used. When d is odd, the second is used. For a given d and an integer $m > d/2$, we write $k_m(x, y) = \phi_s(x \cdot y)$ to denote the corresponding kernel on $\mathbb{S}^d$.

A spherical harmonic expansion of the restricted surface splines can be found in [2, Equations (2.12) & (2.20)]. It is known that, for $m > d/2$, $k_m(x, y) = \sum_\ell \sum_j \widetilde{k}_m(\ell, j) Y_{\ell,j}(x) Y_{\ell,j}(y)$, where

$$\widetilde{k}_m(\ell, j) = C_m \prod_{\nu=1}^{m} \left[\left(\ell + \left(\frac{d-1}{2} \right) \right)^2 - (\nu - 1/2)^2 \right]^{-1}, \quad \text{for } \ell > s. \tag{3.3}$$

When d is odd, this equation holds for all ℓ.

From this formula, it follows that k_m is conditionally positive definite with respect to the space $\Pi_{\lfloor s \rfloor} := \operatorname{span}(Y_{\ell,j} \mid \ell \leq s, j \leq N(d, \ell))$.

A second consequence is that, by a possible slight correction of the spherical harmonic expansion (discussed below), k_m is the fundamental solution for a differential operator of

order $2m$ that is polynomial in Δ:

$$\mathcal{L}_m := C_m \prod_{\nu=1}^{m} \left[\Delta - (\nu - d/2)(\nu + d/2 + 1)\right].$$

We note that when d is odd, the operator $\mathcal{L}_m$ is invertible on $W_2^{2m}(\mathbb{S}^d)$. Indeed, it is nonvanishing on each spherical harmonic $Y_{\ell,m}$.

When d is even, the Fourier coefficients of the kernel follow (3.3) for $\ell > s$ only, but $\mathcal{L}_m$ annihilates spherical harmonics of degree s or less. Indeed, in this case, we can re-index the operator to get:

$$\begin{aligned}
\mathcal{L}_m &= C_m \prod_{\nu=1}^{d/2-1} \left[\Delta - (\nu - d/2)(\nu + d/2 - 1)\right] \prod_{\nu=d/2}^{m} \left[\Delta - (\nu - d/2)(\nu + d/2 - 1)\right] \\
&= C_m \prod_{\nu=1}^{d/2-1} \left[\Delta - (\nu - d/2)(\nu + d/2 - 1)\right] \prod_{J=0}^{m-d/2} \left[\Delta - J(J + d - 1)\right].
\end{aligned}$$

So $\mathcal{L}_m$ annihilates all the spherical harmonics of order up to $s = m - d/2$.

In other words, for sufficiently smooth functions, say $f \in C^{2m}(\mathbb{S}^d)$ represented by the series $f = \sum_{\ell=0}^{\infty} \sum_{m=1}^{N(d,\ell)} \widehat{f}(\ell, j) Y_{\ell,j}$,

$$f(x) = \int_{\mathbb{S}^d} \mathcal{L}_m[f](\alpha)\phi_s(x \cdot \alpha)\mathrm{d}\mu(\alpha) + p_f$$

where we add a spherical harmonic term $p_f = \sum_{\ell=0}^{s} \sum_{j=1}^{N(d,\ell)} \widehat{f}(\ell, j) Y_{\ell,j} \in \Pi_s$ when d is even (when d is odd, $p_f = 0$).

Example 3 (Surface Splines on $SO(3)$). When $\mathbb{M} = SO(3)$, the group of proper rotations of $\mathbb{R}^3$, the eigenvalues of the Laplace–Beltrami operator are $\mu_\ell = \ell(\ell + 1)$ and each eigenvalue is associated with $N(\ell) = (1 + 2\ell)^2$ linearly independent eigenfunctions, called Wigner D-functions and denoted by $(D^\ell_{j,\iota})_{(|j|,|\iota| \le \ell)}$.

For $m \ge 2$ and $s = m - 3/2$, the surface spline kernels are

$$\mathbf{k}_m(x, y) = \left(\sin\left(\frac{\omega(y^{-1}x)}{2}\right)\right)^{2m-3}, \tag{3.4}$$

where $\omega(z)$ is the rotational angle of $z \in SO(3)$. The corresponding Wigner D-function expansion $\mathbf{k}_m(x, y) = \sum_\ell \sum_{j,\iota} \widetilde{\mathbf{k}}_m(\ell, j, \iota) D^{(\ell)}_{j,\iota}(x) D^{(\ell)}_{j,\iota}(y)$ is discussed in [17, Lemma 2], where it is shown that for some $C_m \ne 0$,

$$\widetilde{\mathbf{k}}_m(\ell, j, \iota) = C_m \prod_{\nu=-(m-1)}^{m-1} \left[\ell + \nu + \frac{1}{2}\right]^{-1}.$$

Thus, $\mathbf{k}_m$ is conditionally positive definite with respect to the space $\Pi_{m-2} = \mathrm{span}\{D^\ell_{j,\iota} \mid \ell \le m - 2,\ |j|, |\iota| \le \ell\}$.

It also follows (from [17, Lemma 3]) that $\mathbf{k}_m$ is the fundamental solution for the differential operator of order $2m$ having the form:

$$\mathcal{L}_m := C_m \prod_{\nu=0}^{m-1} \left[\Delta - (\nu^2 - 1/4)\right]$$

in the sense that for $f \in C^{2m}$, $f = \sum_{\ell=0}^{\infty} \sum_{|j|,|\iota| \le \ell} \widehat{f}(\ell, j, \iota) D^{\ell}_{j,\iota}$ the formula

$$f(x) = \int_{SO(3)} \mathcal{L}_m[f](\alpha)\mathbf{k}_m(x,\alpha)\mathrm{d}\mu(\alpha)$$

holds true.

3.3 Polyharmonic and related kernels

The kernels we wish to treat are fundamental solutions of differential operators that are polynomial in the Laplace–Beltrami operator, or are directly related to them. Since, on a compact Riemannian manifold Δ is a self adjoint operator with a countable sequence of nonnegative eigenvalues $\lambda_j \le \lambda_{j+1}$ having $+\infty$ as the only accumulation point, we can express the kernel in terms of the associated eigenfunctions $\Delta\phi_j = \lambda_j \phi_j$. We now make this clear with a formal definition.

Definition 3.2. *Let $m \in \mathbb{N}$ such that $m > d/2$. We say that the kernel $k_m : \mathbb{M} \times \mathbb{M} \to \mathbb{R}$ is polyharmonic if the following hold:*

1. *There exists a polynomial $Q(x) = \sum_{\nu=0}^{m} c_\nu x^\nu$ in $\Pi_m(\mathbb{R})$, with the highest order coefficient $c_m > 0$, so that $Q(x) > 0$ for all x sufficiently large. Let the corresponding differential operator of order $2m > d$ be given by*

$$\mathcal{L}_m = \sum_{\nu=0}^{m} c_\nu \Delta^\nu = Q(\Delta),$$

   and let $\mathcal{J} \subset \mathbb{N}$ be a finite set that includes all j for which the eigenvalue $Q(\lambda_j)$ of $\mathcal{L}_m$ satisfies $Q(\lambda_j) \le 0$. (In addition to this finite set, $\mathcal{J}$ may also include a finite number j's for which $Q(\lambda_j) > 0$.)

2. *The kernel has the eigenfunction expansion $k_m(x,y) = \sum_{j\in\mathbb{N}} \tilde{k}_m(j)\psi_j(x)\psi_j(y)$, with coefficients $\tilde{k}_m(j) = 1/Q(\lambda_j)$ for $j \notin \mathcal{J}$. (On $\mathcal{J}$, $\tilde{k}_m(j)$ can assume arbitrary values.)*

It follows immediately from this definition that k_m is conditionally positive definite with respect to the finite dimensional space $\Pi_{\mathcal{J}} = \mathrm{span}_{j\in\mathcal{J}}\,\varphi_j$. Another consequence is that, for $f \in C^\infty$,

$$f(x) = \int_{\mathbb{M}} \mathcal{L}_m[f - p_f](\alpha) k_m(x,\alpha)\mathrm{d}\mu(\alpha) + p_f \tag{3.5}$$

where $p_f = \sum_{j\in\mathcal{J}} \mathrm{proj}_j f$ is the orthogonal projection onto $\Pi_{\mathcal{J}}$.

As previously stated, the interpolation operator $I_{\Xi,k_m,\mathcal{J}}$ produces the minimizer of the seminorm $\|u\|_{k_m,\mathcal{J}}$. Since $k_m(x,y) = \sum_{j\notin\mathcal{J}} \tilde{k}_m(j)\varphi_j(x)\varphi_j(y)$ and, for $j \notin \mathcal{J}$,

$$\tilde{k}_m(j) = Q(\lambda_j)^{-1} = \left(\sum_{\nu=0}^{m} c_\nu (\lambda_j)^\nu\right)^{-1},$$

which is the inverse symbol of $\mathcal{L}_m$, it follows from (3.1) that

$$\|u\|_{k_m,\mathcal{J}}^2 = \sum_{j\notin\mathcal{J}} \frac{|\hat{u}(j)|^2}{\tilde{k}(j)} = \langle \mathcal{L}_m u, u\rangle_{L_2(\mathbb{M})} - \sum_{j\in\mathcal{J}} Q(\lambda_j)|\hat{u}(j)|^2.$$

This relation connects the norm $\|u\|_{k_m,\mathcal{J}}$ with the quadratic form $\langle \mathcal{L}_m u, u\rangle_{L_2(\mathbb{M})}$. The goal of the next section is to study this quadratic form.

4 Operators and quadratic forms

Of the two quadratic forms considered, the one derived from the native space seminorm: $\|u\|_{k_m,\mathcal{J}}^2$, and the one derived from the operator $[u]^2 := \langle \mathcal{L}_m u, u\rangle_{L_2(\mathbb{M})}$, the latter has much to offer from the point of view of analysis, but the former is tied to the variational problem satisfied by the kernel interpolants. The object of this section is to attain a better understanding of $[u]^2$.

To this end, we seek an analogue of the bilinear form $\langle \mathcal{L}u, v\rangle_{L_2(\mathbb{M})}$ – one that is defined on measurable subsets of $\mathbb{M}$. A reasonable goal would be to find a form that is comparable to the corresponding Sobolev form $\sum_{j=0}^{m}\langle u, v\rangle_{j,\Omega}$, where

$$\langle u, v\rangle_{j,\Omega} := \int_\Omega \langle \nabla^j u, \nabla^j v\rangle_{g,x} \mathrm{d}\mu(x).$$

This is the bilinear form used to define the Sobolev space inner product: (2.8) of Definition 2.1 for $\Omega \subset \mathbb{M}$.

The rest of this section is structured as follows. In 4.1 we demonstrate that on a wide class of manifolds, the elliptic operator composed of covariant and contravariant derivatives, which is at the heart of [16], is a polynomial in Δ and, conversely, the Laplace–Beltrami operator has an expansion in terms of these elliptic operators. This permits us immediately to classify the Sobolev kernels on spheres (as well-known kernels of a type studied in [14, 22]) and to give concrete approximation results for them. In 4.2 we present analogues to the bilinear form generated by $\mathcal{L}_m$ on measurable subsets. Using this, we demonstrate that this bilinear form behaves like a norm for functions with many zeros.

4.1 The Laplace–Beltrami operator and the covariant derivative

Simply considering $\langle \mathcal{L}u, v\rangle_{L_2(\Omega)}$ for measurable subsets $\Omega \subset \mathbb{M}$ is not suitable, since there will be many functions for which $\mathcal{L}u$ may vanish on Ω. This is true even on $\mathbb{R}^d$ when $\mathcal{L} = \Delta^m$. In this case there are many polyharmonic functions (and even harmonic functions!) on a a

given subdomain Ω that may have nonzero Sobolev norms, despite the fact that they are in the kernel of $\mathcal{L}$.

Guided by the observation that on $\mathbb{R}^d$, $\int_{\mathbb{R}^d} v\Delta^m u = \int_{\mathbb{R}^d}\langle \nabla^m v \nabla^m u\rangle$ holds for test functions u, v, we first attempt to compare Δ^m, the principle part of $\mathcal{L}$, to $(\nabla^m)^*\nabla^m$. It is important to stress that $(\nabla^m)^*$ means the adjoint to ∇^m, in the $L_2(\mathbb{M})$ inner product, as defined in (2.7). [2] To this end, we make the following assumption

Assumption 4.1. *For all $k \in \mathbb{N}$, there exists a real polynomial p_{k-1} of degree $k-1$, such that*

$$(\nabla^k)^*\nabla^k = (-1)^k\Delta^k + p_{k-1}(\Delta).$$

A class of Riemannian manifolds that satisfy this are the *two-point homogeneous spaces* [18], both compact and non-compact. A manifold $\mathbb{M}$ is homogeneous if $\mathbb{M} = G/K$, where G is a Lie group and K is a Lie subgroup of G. *Two-point* homogeneous means that for any two pairs of points p, q and p', q' such that the distances $d(p,q) = d(p',q')$ there is an isometry $\Phi \in G$ such that $p' = \Phi(p)$ and $q' = \Phi(q)$. These manifolds[3] have been completely classified (see [18, p. 167 & p. 177] for lists), and include $\mathbb{S}^d$ and the real projective spaces P^d. (The rotation group $SO(3) = \mathsf{P}^3$.)

Lemma 4.2. *Let $\mathbb{M}$ be a two-point homogeneous space. Then $\mathbb{M}$ satisfies Assumption 4.1.*

Proof. The proof proceeds in two steps. The first is showing that if $\Phi : \mathbb{M} \to \mathbb{M}$ is a diffeomorphism that is also an isometry (i.e., preserves distances), then the operator $D := (\nabla^k)^*\nabla^k$ is invariant in the sense that for any smooth function $f : \mathbb{M} \to \mathbb{R}$, $Df = (D(f\circ\Phi))\circ\Phi^{-1}$. We will follow a technique used in [18, Proposition 2.4, p. 246]. Let $(\mathfrak{U}, \phi)$ be a local chart, with coordinates $x^j = \phi^j(p)$, $j = 1, \ldots, n$ for $p \in \mathfrak{U}$. Since Φ is a diffeomorphism, $(\Phi(\mathfrak{U}), \phi\circ\Phi^{-1})$ is also a local chart. Let $\psi = \phi\circ\Phi^{-1}$, and use the coordinates $y^j = \psi^j(q)$ for $q \in \Phi(\mathfrak{U})$. The choice of coordinates has the effect of assigning the *same* point in $\mathbb{R}^n$ to p and q, provided $q = \Phi(p)$ – i.e., $x^j(p) = y^j(q)$. Thus, relative to these coordinates the map Φ is the identity, and consequently, the two tangent vectors $(\frac{\partial}{\partial y^j})_q \in T_q\mathbb{M}$ and $(\frac{\partial}{\partial x^j})_p \in T_p\mathbb{M}$ are related via

$$\left(\frac{\partial}{\partial y^j}\right)_{\Phi(p)} = d\Phi_p\left(\frac{\partial}{\partial x^j}\right)_p.$$

So far, we have only used the fact that Φ is a diffeomorphism. The map Φ being in addition an isometry then implies that

$$\left\langle \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^k}\right\rangle_{\Phi(p)} = \left\langle d\Phi_p\left(\frac{\partial}{\partial x^j}\right), d\Phi_p\left(\frac{\partial}{\partial x^k}\right)\right\rangle_{\Phi(p)} = \left\langle \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right\rangle_p.$$

The expression on the left is the metric tensor at $\Phi(p)$, $g^{jk}(y)$, and on the right, the metric tensor at p, $g^{jk}(x)$. The equation above implies that, as *functions* of y and x, $g^{jk}(y) = g^{jk}(x)$.

[2] and not with respect to $L_2(\Omega)$ – e.g., even though $\Delta^m = (-1)^m(\nabla^m)^*\nabla^m$ holds on $\mathbb{R}^d$, it is not the case that $(-1)^m\int_\Omega v\Delta^m u = \int_\Omega \nabla^m v\nabla^m u$ for subsets $\Omega \subset \mathbb{R}^d$.

[3] We note that they have also appeared in other approximation theory literature. See, e.g., [3, 20]

This means that the expressions for the Christoffel symbols, covariant derivatives and various expressions formed from them will, as functions, be the same. Since the operator $D = (\nabla^k)^*\nabla^k$ is constructed from such objects, it follows that $Df = (D(f \circ \Phi)) \circ \Phi^{-1}$, and so D is invariant.

The second step makes use of two-point homogeneity. Since $(\nabla^k)^*\nabla^k$ is invariant under every isometry Φ in G, applying [18, Proposition 4.11, p. 288] yields the result that $(\nabla^k)^*\nabla^k$ is a polynomial in the Laplace–Beltrami operator: $(\nabla^k)^*\nabla^k = a_{k+1}\Delta^k + a_k\Delta^{k-1} + \cdots a_0$. Comparing terms in the highest order derivatives involved in coordinate expressions for both sides shows that $a_{k+1} = (-1)^k$. □

Induction ensures that

$$\Delta^k - (-1)^k(\nabla^k)^*\nabla^k = c_{k-1}(\nabla^{k-1})^*\nabla^{k-1} + c_{k-2}(\nabla^{k-2})^*\nabla^{k-2} \cdots + c_0.$$

From this we have the following.

Lemma 4.3. *Suppose $\mathbb{M}$ is a Riemannian manifold satisfying Assumption 4.1. If $Q(x) = c_m x^m + \cdots + c_0$ is a (real) polynomial of degree m, then there exist real numbers a_j, with $a_m = c_m$, so that*

$$Q(\Delta) = \sum_{j=0}^{m} a_j (\nabla^j)^*\nabla^j.$$

Conversely, for any constants b_j, there is a real polynomial p for which the operators $p(\Delta)$ and $\sum_{j=0^n} b_j(\nabla^j)^*\nabla^j$ coincide.

An immediate consequence is that the Sobolev kernels $\kappa_{m,\mathbb{M}}$ considered in [16] and [15] are Green's functions for operators of the form $Q(\Delta)$, with Q a real polynomial of degree m.

We note, furthermore, that because the lead coefficient c_m of Q is assumed positive (see Definition 3.1), we have that $a_m > 0$.

4.2 Connecting the quadratic form to the Sobolev norm

The benefit of Lemma 4.3 is that we can use it to obtain useful local versions of the form $\langle \mathcal{L}u, v\rangle_{L_2(\mathbb{M})}$. In particular, we consider, for $\mathcal{L}_m = Q(\Delta)$, coefficients $a_0, \dots, a_m$ as guaranteed by Lemma 4.3. When $\Omega \subset \mathbb{M}$, the form $\langle \sum_{j=0}^m a_j(\nabla^j)^*\nabla^j u, v\rangle_{L_2(\Omega)} = \sum_{j=0}^m a_j\langle u, v\rangle_{j,\Omega}$ is the local version of $\langle \mathcal{L}u, v\rangle_{L_2(\mathbb{M})}$. Indeed, we have

$$[u]_{m,\Omega}^2 := \int_\Omega \beta(u,u)_x \mathrm{d}\mu$$

where $\beta(\cdot,\cdot)_x : C^\infty \times C^\infty \to \mathbb{R}$ is the bilinear form $\beta(u,v)_x = \sum_{j=0}^m a_j\langle \nabla^j u, \nabla^j v\rangle_x$. Clearly for

$$\overline{a} := \max_{j\le m} |a_j| \qquad \text{and} \qquad \underline{a} = \max_{j\le m-1} |a_j| \tag{4.1}$$

we have

$$a_m\langle \nabla^m u, \nabla^m u\rangle_x - \underline{a}\sum_{j=0}^{m-1}\langle \nabla^j u, \nabla^j u\rangle_x \le \beta(u,u)_x \le \overline{a}\sum_{j=0}^{m}\langle \nabla^j u, \nabla^j u\rangle_x. \tag{4.2}$$

If we integrate over a region $\Omega \subset \mathbb{M}$, we obtain

$$a_m|u|^2_{W_2^m(\Omega)} - \underline{a}\|u\|^2_{W_2^{m-1}(\Omega)} \le [u]^2_{m,\Omega} := \int_\Omega \beta(u,u)_x \mathrm{d}\mu(x) \le \overline{a}\|u\|^2_{W_2^m(\Omega)}.$$

Now if u vanishes on a sufficiently dense set $X_0 \subset \Omega$, then a corollary of the "zeros" estimate Theorem A.11, given in Section A, will imply that $a_m|u|^2_{W_2^m(\Omega)} - \underline{a}\|u\|^2_{W_2^{m-1}(\Omega)} \ge \varepsilon\|u\|^2_{W_2^m(\Omega)}$ where ε depends on a_m, $\underline{a}$, properties of X_0 and the boundary of Ω, but nothing else. The two most important types of subset Ω, for our purposes, are annuli $\mathbf{a}$ and complements of balls $\mathbf{b}^c$.

Annuli. In this case we consider an annulus $\mathbf{a} = B(p_0,R) \setminus B(p_0,R-t)$ with outer radius $R < \frac{1}{2}\mathrm{r}_{\mathbb{M}}$ and we apply Corollary A.16 to a function u that vanishes on a set X satisfying $h = h(X,\mathbf{a}) \le \gamma t$. If, in addition, $h \le \sqrt{\frac{a_m}{2\Lambda\overline{a}}}$, we have $\|u\|^2_{W_2^{m-1}(\mathbf{a})} \le \Lambda h^2 |u|^2_{W_2^m(\mathbf{a})} \le \frac{a_m}{2\underline{a}}|u|^2_{W_2^m(\mathbf{a})}$ and, simultaneously, $\frac{a_m}{4}\|u\|^2_{W_2^m(\mathbf{a})} \le \frac{a_m}{2}|u|^2_{W_2^m(\mathbf{a})}$, so $a_m|u|^2_{W_2^m(\Omega)} - \underline{a}\|u\|^2_{W_2^{m-1}(\Omega)} \ge \frac{a_m}{4}\|u\|^2_{W_2^m(\Omega)}$ follows and

$$\frac{a_m}{4}\|u\|^2_{W_2^m(\mathbf{a})} \le \int_{\mathbf{a}} \beta(u,u)\mathrm{d}\mu \le \overline{a}\,\|u\|^2_{W_2^m(\mathbf{a})}. \tag{4.3}$$

(Note that h must be chosen to be less than $\sqrt{\frac{a_m}{2\Lambda\overline{a}}}$, as well as γt).

Complements of balls We consider the punctured manifold $\mathbf{b}^c = \mathbb{M} \setminus \mathbf{b}(p_0,R)$ with outer radius $R < \frac{1}{2}\mathrm{r}_{\mathbb{M}}$. We apply Corollary A.17 to a function u that vanishes on a set X satisfying $h = h(X,\mathbf{a}) \le \gamma t$. By picking $h < \sqrt{\frac{a_m}{2\Lambda\overline{a}}}$,

$$\frac{a_m}{4}\|u\|^2_{W_2^m(\mathbf{b}(p,r)^c)} \le \int_{\mathbf{b}(p,r)^c} \beta(u,u)\mathrm{d}\mu \le \overline{a}\|u\|^2_{W_2^m(\mathbf{b}(p,r)^c)}. \tag{4.4}$$

follows. (Note that in this case, h must be less than h_0 and $\sqrt{\frac{a_m}{2\Lambda\overline{a}}}$, but that it can be chosen independently of r. In Lemma 5.1 we refer to this critical value, the minimum of h_0 and $\sqrt{\frac{a_m}{2\Lambda\overline{a}}}$, as H_0.)

5 The Lagrange function

We wish to uniformly bound the Lagrange function $\chi_\xi(x)$ and establish its rate of decay as x moves away from its center ξ. There are two cases that we will consider.

The first is the special case that involves interpolation by a polyharmonic kernel k_m (cf. Definition 3.2) that is conditionally positive definite with respect to a space Π annihilated by the operator $\mathcal{L}_m$. This case is significant because the rate of decay is exponential (cf. Theorem 5.3). It includes the restricted surface splines on $\mathbb{S}^d$ discussed in Example 2, for d even.

The second case is the general one, where we do not assume any annihilation properties concerning the space Π that is to be reproduced. This case includes the surface splines in odd dimensions. The decay rate in this case is algebraic, rather than exponential.

These results are similar to ones for the case of a lattice in $\mathbb{R}^d$ [5]. The restricted surface splines defined in (3.2) have Lagrange functions that decay exponentially, for d even, but only algebraically for d odd. For d odd, the lattice case has an additional family of polyharmonic splines with exponential decay. We conjecture that this exponential decay holds for odd-dimensional spheres, and that we have obtained only algebraic decay is simply an artifact of the proof.

Notation	Constant	Introduce in ...
a_m	(Positive) lead coefficient of the polynomial $Q(x)$	Lemma 4.3
$\overline{a}$	maximum coefficient of $Q(x)$ (in absolute value)	(4.1)
$\underline{a}$	maximum coefficient of $Q(x) - a_m x^m$	(4.1)
C_Q	$\ell_1(\mathbb{R}^{\mathcal{J}})$ norm of eigenvalues of $\mathcal{L}_m\|_{\Pi_{\mathcal{J}}}$	(5.5)
$\mathrm{r}_{\mathbb{M}}$	injectivity radius	Section 2
Γ_1, Γ_2	constants of metric equivalence from Exp	(2.3)
c_1, c_2	constants of metric equivalence for Sobolev spaces	Lemma 2.2
Λ	constant for zeros lemma for annuli	(A.20)
h_0	threshold h level for the zeros lemma	(A.19)
H_0	threshold h level for results of 5.1	Lemma 5.1
H_1	threshold h level for results of 5.2	Lemma 5.4

Table 2: Constants frequently used in Section 5. The first four constants are related to the elliptic operator $\mathcal{L}_m = Q(\Delta)$. The final seven are geometric constants depending on $\mathbb{M}$.

5.1 $\mathcal{L}_m$ annihilates $\Pi_{\mathcal{J}}$

We first consider the special case where k_m satisfies (3.5), with an operator $\mathcal{L}_m = Q(\Delta)$ for which $\widetilde{k}_m(j) = (Q(j))^{-1} > 0$ for $j \notin \mathcal{J}$ and $\mathcal{L}_m \phi_j = 0$ for $j \in \mathcal{J}$. In other words, k_m is conditionally positive definite with respect to $\Pi_{\mathcal{J}}$, and $\mathcal{L}_m \Pi_{\mathcal{J}} = 0$. This the case for Example 2 for surface splines on even dimensional spheres.

In this case, the native space seminorm (3.1) is precisely the quadratic form derived from the operator, namely

$$\|\!|u|\!\|^2_{k_m,\mathcal{J}} = \langle \mathcal{L}_m u, u\rangle_{L_2(\mathbb{M})} = [u]^2_{k_m,\mathbb{M}}.$$

The more general case is considered in the next section, although the basic elements are present here.

We begin by observing that if Ξ is sufficiently dense, with $h \le \min(h_0, \sqrt{\frac{a_m}{2\Lambda\overline{a}}})$, then by (4.4) it is possible to estimate the norm of the Lagrange function by comparing it to a bump ϕ_ξ with $\phi_\xi \circ \mathrm{Exp}_\xi(x) = \sigma(|x|/q)$. We note that this bump is 1 at ξ and vanishes on the rest of Ξ, thus it interpolates χ_ξ on Ξ and has a smaller native space seminorm.

$$\frac{a_m}{4}\|\chi_\xi\|^2_{W_2^m(\mathbb{M})} \le \|\!|\chi_\xi|\!\|^2_{k_m,\mathcal{J}} \le \|\!|\phi_\xi|\!\|^2_{k_m,\mathcal{J}} \le \overline{a}\|\phi_\xi\|^2_{W_2^m(\mathbb{M})} \le C\overline{a}q^{d-2m}. \tag{5.1}$$

The final inequality follows from Lemma 2.2, and a direct computation of $\|\sigma(|\cdot|/q)\|_{W_2^{2m}(\mathbb{R}^d)}$.

The main result, the near-exponential decay of the Lagrange functions, now is a consequence of an argument developed in [16] but given here in a somewhat different, streamlined form. First we prove a lemma showing that a fraction of the seminorm of the Lagrange function χ_ξ taken over the punctured manifold $\mathbf{b}(\xi, r)^c$ resides in a narrow annular region around the circle $\mathrm{dist}(x, \xi) = r$.

Lemma 5.1. *Suppose $\mathbb{M}$ is a d-dimensional compact Riemannian manifold satisfying Assumption 4.1. Suppose further that $m > d/2$, k_m satisfies Definition 3.2 and that $\mathcal{L}_m$ annihilates the space $\Pi_{\mathcal{J}}$. Then there is a constant $K > 0$, depending only on m and $\mathbb{M}$ so that the following holds. If Ξ is sufficiently dense, meaning that*

$$h < H_0 := \min\left(h_0, \sqrt{\frac{a_m}{2\Lambda \overline{a}}}\right)$$

and if $\mathfrak{a} = \mathbf{b}(p, r) \setminus \mathbf{b}(p, r-t)$ is an annulus of outer radius $r < \mathrm{r}_\mathbb{M}$ and sufficient width t, so that $4h/h_0 \le t$, then the Lagrange functions for interpolation by k_m satisfy

$$\|\chi_\xi\|^2_{W_2^m(\mathbf{b}(p,r-t)^c)} \le K \|\chi_\xi\|^2_{W_2^m(\mathbf{b}(p,r)\setminus\mathbf{b}(p,r-t))}.$$

Proof. Since χ_ξ minimizes the native space seminorm we have $[\chi_\xi]^2_{k_m,\mathbb{M}} \le [\phi_\xi \chi_\xi]^2_{k_m,\mathbb{M}}$ for any function ϕ_ξ equaling 1 at ξ. If ϕ_ξ is a C^∞ cut-off, equaling 1 in the ball $\mathfrak{b} = \mathbf{b}(p, r-t)$ and vanishing outside of the ball $\mathfrak{b} \cup \mathfrak{a}$, then

$$[\chi_\xi]^2_{k_m,\mathfrak{b}} + [\chi_\xi]^2_{k_m,\mathfrak{b}^c} \le [\chi_\xi]^2_{k_m,\mathfrak{b}} + [\phi_\xi \chi_\xi]^2_{k_m,\mathfrak{a}}$$

By (4.4) and (4.3)

$$\frac{a_m}{4} \|\chi_\xi\|^2_{W_2^m(\mathfrak{b}^c)} \le [\chi_\xi]^2_{k_m,\mathfrak{b}^c} \le [\phi_\xi \chi_\xi]^2_{k_m,\mathfrak{a}} \le \overline{a} \|\phi_\xi \chi_\xi\|^2_{W_2^m(\mathfrak{a})}.$$

The result follows with $K = 4\overline{a}K'/a_m$, where the constant K' is introduced in Lemma 5.2, which we prove below. ◻

Lemma 5.2. *Assume the manifold $\mathbb{M}$, the kernel k_m, the set of centers Ξ and the annulus $\mathfrak{a} \subset \mathbb{M}$ satisfy the conditions of Lemma 5.1. If ϕ_ξ is a smooth "bump" function, satisfying*

$$\phi_\xi \circ \mathrm{Exp}_\xi(x) = \sigma\left(\frac{1}{t}\mathrm{dist}(\mathrm{Exp}_\xi(x), \mathrm{Exp}_\xi(0)) - \frac{2t-r}{t}\right) = \sigma\left(\frac{|x|}{t} + \frac{2t-r}{t}\right) \tag{5.2}$$

with $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ a C^∞, non-increasing cutoff function equaling 1 on $[0,1]$ and 0 on $[2, \infty)$, then

$$\|\phi_\xi \chi_\xi\|_{W_2^m(\mathfrak{a})} \le K' \|\chi_\xi\|_{W_2^m(\mathfrak{a})}$$

where K' depends only on $\mathbb{M}$, m and the choice of cutoff σ.

Proof. We follow the proof of [16, Lemma 4.3]. Let $\widetilde{\chi_\xi}(x) = \chi_\xi \circ \mathrm{Exp}_\xi$. By using the metric equivalence guaranteed by Lemma 2.2, we can estimate $\|\phi_\xi\chi_\xi\|^2_{W_2^m(\mathfrak{a})}$ by

$$\begin{aligned}
\|\phi_\xi\chi_\xi\|^2_{W_2^m(\mathfrak{a})} &\leq c_2^2\int_{\mathbb{R}^d}\sum_{|\alpha|\leq m}\left|D^\alpha\left[\sigma\left(\frac{|x|}{t}+\frac{2t-r}{t}\right)\widetilde{\chi_\xi}(x)\right]\right|^2 \mathrm{d}x\\
&\leq c_2^2 C\sum_{j=0}^m t^{2(j-m)}\int_{B(0,r)\setminus B(0,r-t)}\sum_{|\alpha|=j}|D^\alpha\widetilde{\chi_\xi}(x)|^2\,\mathrm{d}x\\
&\leq \left(\frac{c_2}{c_1}\right)^2 C\sum_{j=0}^m t^{2(j-m)}\|\chi_\xi\|^2_{W_2^j(\mathfrak{a})} \leq C\left(\frac{c_2}{c_1}\right)^2\Lambda\sum_{j=0}^m\left(\frac{h}{t}\right)^{2(m-j)}\|\chi_\xi\|^2_{W_2^m(\mathfrak{a})}.
\end{aligned}$$

and $K' = C(\frac{c_2}{c_1})^2\Lambda\sum_{j=0}^m(h_0/4)^{2(m-j)}$. The second inequality follows from the product rule, and C is a constant depending only on m, d and σ. The third inequality is Lemma 2.2 again, and the final inequality is the zeros lemma for annuli, Corollary A.16. □

At this point, we can follow the example of [16, Section 4]

Theorem 5.3. *Suppose that $\mathbb{M}$ is a compact d-dimensional Riemannian manifold satisfying Assumption 4.1. Suppose further that $m > d/2$ and that k_m satisfies Definition 3.2 and that $\mathcal{L}_m$ annihilates the space $\Pi_{\mathcal{J}}$. There exist positive constants h_0, ν and C, depending only on m, $\mathbb{M}$ and the operator $\mathcal{L}_m$ so that if the set of centers Ξ is quasiuniform with mesh ratio ρ and has density $h \leq H_0$ then the Lagrange functions for interpolation by k_m satisfy*

$$|\chi_\xi(x)| \leq C\rho^{m-d/2}\exp\left(-\frac{\nu}{h}\min\bigl(d(x,\xi),\mathrm{r}_{\mathbb{M}}\bigr)\right). \tag{5.3}$$

Furthermore, for any $0 < \epsilon \leq 1$, there is a constant C depending only on $m, \mathbb{M}, \rho$ and ϵ, so that the Lagrange functions satisfy

$$|\chi_\xi(x) - \chi_\xi(y)| \leq C\left(\frac{d(x,y)}{q}\right)^\epsilon. \tag{5.4}$$

Proof. Set $t = 4h/h_0 =: \gamma h$, and note that for $t \leq r \leq \mathrm{r}_{\mathbb{M}}$, Lemma 5.1 implies that

$$\|\chi_\xi\|^2_{W_2^m(\mathbf{b}(\xi,r)^c)} \leq \epsilon\|\chi_\xi\|^2_{W_2^m(\mathbf{b}(\xi,r-t)^c)},$$

with $\epsilon = (K-1)/K$. Letting $n := \lfloor r/t\rfloor$, we have

$$\begin{aligned}
\|\chi_\xi\|^2_{W_2^m(\mathbf{b}(\xi,r)^c)} &\leq \epsilon^n\|\chi_\xi\|^2_{W_2^m(\mathbb{M})}\\
&\leq \epsilon^{-1}e^{(\log\epsilon)r/t}\|\chi_\xi\|^2_{W_2^m(\mathbb{M})} \leq \epsilon^{-1}e^{-\nu r/h}\|\chi_\xi\|^2_{W_2^m(\mathbb{M})}\\
&\leq Ce^{-\nu r/h}q^{2m-d}
\end{aligned}$$

with $\nu := -\gamma\log\epsilon$. Since $\epsilon = \frac{K}{K+1} < 1$, it follows that $\nu > 0$. The final inequality follows from (5.1).

The bound (5.3) follows from the observation that $\chi_\xi(x)$ can be estimated using Theorem A.11. The intersection $\mathbf{b}(x,t)\cap\mathbf{b}(\xi,\mathrm{dist}(x,\xi))^c$ is contained the ball $\mathbf{b}(x,R)$, since $t<R<\frac{1}{2}\mathrm{r}_{\mathbb{M}}$. Because geodesic spheres are smooth hypersurfaces whose intersection is nontangential, The intersection $\mathbf{b}(x,t)\cap\mathbf{b}(\xi,\mathrm{dist}(x,\xi))^c$ is a Lipschitz domain contained in $\mathbf{b}(x,R)$. Moreover, $h\le\gamma t$. Thus, Theorem A.11 applies, giving us

$$|\chi_\xi(x)|\le Ch^{m-d/2}\|\chi_\xi\|_{W_2^m(\mathbf{b}(x,t)\cap\mathbf{b}(\xi,\mathrm{dist}(x,\xi))^c)}\le Ch^{m-d/2}\|\chi_\xi\|_{W_2^m\left(\mathbf{b}\left(\xi,d(x,\xi)\right)^c\right)}$$

for $h<\gamma t$. Similarly, the estimate in (5.4) follows from Corollary A.15. □

5.2 General Case

In this case, the native space seminorm (3.1) and the quadratic form induced by the operator, differ by some low order terms:

$$\|u\|_{k_m,\mathcal{J}}^2=\langle\mathcal{L}_m u,u\rangle_{L_2(\mathbb{M})}-\sum_{j\in\mathcal{J}}Q(\lambda_j)|\langle u,\varphi_j\rangle|^2=[u]_{k_m,\mathbb{M}}^2-\sum_{j\in\mathcal{J}}Q(\lambda_j)|\langle u,\varphi_j\rangle|^2.$$

Because of the orthonormality of φ_j, we have $|\langle u,\varphi_j\rangle|^2\le\|u\|_{L_2(\mathbb{M})}^2$. Setting

$$C_Q:=\sum_{j\in\mathcal{J}}|Q(\lambda_j)| \tag{5.5}$$

(this is the $\ell_1(\mathbb{R}^{\mathcal{J}})$ norm of the spectrum of the operator $\mathcal{L}_m$ restricted to $\Pi_{\mathcal{J}}$) we note that, by Corollary A.13, if u vanishes on a sufficiently dense set, then the lower order terms are controlled

$$\sum_{j\in\mathcal{J}}|Q(\lambda_j)||\langle u,\varphi_j\rangle|^2\le C_Q h^{2m}\|u\|_{W_2^m(\mathbb{M})}^2.$$

Indeed it follows that $\frac{a_m}{8}\|u\|_{W_2^m}^2\le\|u\|_{k_m,\mathcal{J}}^2$ when h is chosen small enough that $C_Q h^{2m}\le\frac{a_m}{8}$.

This allows us to provide a basic estimate for the Lagrange function, similar to (5.1). In this case

$$\frac{a_m}{8}\|\chi_\xi\|_{W_2^m(\mathbb{M})}^2\le\|\chi_\xi\|_{k_m,\mathcal{J}}^2\le\|\phi_\xi\|_{k_m,\mathcal{J}}^2\le(\overline{a}+C_Q h^{2m})\|\phi_\xi\|_{W_2^m(\mathbb{M})}\le C\overline{a}q^{d-2m}. \tag{5.6}$$

Lemma 5.4. *Suppose $\mathbb{M}$ is a d-dimensional compact Riemmanian manifold satisfying Assumption 4.1. Suppose further that $m>d/2$ and that k_m satisfies Definition 3.2. Then there is a constant $K>0$, depending only on m and $\mathbb{M}$ so that the following holds. If Ξ is sufficiently dense, meaning that*

$$h<H_1:=\min\left(H_0,\sqrt[2m]{\frac{a_m}{8C_Q}}\right)=\min\left(h_0,\sqrt{\frac{a_m}{2\Lambda\overline{a}}},\sqrt[2m]{\frac{a_m}{8C_Q}}\right)$$

and if $\mathfrak{a}=\mathbf{b}(p,r)\setminus\mathbf{b}(p,r-t)$ is an annulus of outer radius $r<\mathrm{r}_{\mathbb{M}}$, satisfying, in addition,

$$\|\chi_\xi\|_{W_2^m(\mathbf{b}(p,r)^c)}\ge C_0 h^{2m}\|\chi_\xi\|_{W_2^m(\mathbb{M})}$$

(for a constant C_0 depending only on m, $\mathbb{M}$, k_m, and $\mathcal{J}$ which we define in the proof) and sufficient width t, so that $h \le \gamma t$, then the Lagrange functions for interpolation by k_m with auxiliary space $\Pi_{\mathcal{J}}$ satisfy

$$\|\chi_\xi\|^2_{W_2^m(\mathbf{b}(p,r-t)^c)} \le K\|\chi_\xi\|^2_{W_2^m(\mathbf{b}(p,r)\setminus\mathbf{b}(p,r-t))}.$$

Proof. Since χ_ξ minimizes the native space seminorm, we have

$$[\chi_\xi]^2_{k_m,\mathbb{M}} \le [\phi_\xi\chi_\xi]^2_{k_m,\mathbb{M}} - \sum Q(\lambda_j)\left(\left|\int_{\mathbb{M}}\phi_\xi(x)\chi_\xi(x)\overline{\varphi_j(x)}\mathrm{d}\mu(x)\right|^2 - \left|\int_{\mathbb{M}}\chi_\xi(x)\overline{\varphi_j(x)}\mathrm{d}\mu(x)\right|^2\right)$$

for a cut-off ϕ_ξ equaling 1 in the ball $\mathfrak{b} = \mathbf{b}(p, r-t)$ and vanishing outside of the ball $\mathfrak{b}(p,r) = \mathfrak{b}\cup\mathfrak{a}$. Using the sum of squares factorization $|A|^2 - |B|^2 = \Re\big[(A-B)(\overline{A}+\overline{B})\big]$, we may write

$$\begin{aligned}
&\left|\int_{\mathbb{M}}\phi_\xi(x)\chi_\xi(x)\overline{\varphi_j(x)}\mathrm{d}\mu(x)\right|^2 - \left|\int_{\mathbb{M}}\chi_\xi(x)\overline{\varphi_j(x)}\mathrm{d}\mu(x)\right|^2\\
&\quad= \Re\left[\left(\int_{\mathbb{M}}\big(\phi_\xi(x)-1\big)\chi_\xi(x)\overline{\varphi_j(x)}\mathrm{d}\mu(x)\right)\times\left(\int_{\mathbb{M}}\big(\phi_\xi(x)+1\big)\chi_\xi(x)\varphi_j(x)\mathrm{d}\mu(x)\right)\right]\\
&\quad= \Re\left[\left(\int_{\mathfrak{a}}\big(\phi_\xi(x)\big)\chi_\xi(x)\overline{\varphi_j(x)}\mathrm{d}\mu(x) - \int_{\mathfrak{b}^c}\chi_\xi(x)\overline{\varphi_j(x)}\mathrm{d}\mu(x)\right)\right.\\
&\qquad\left.\times\left(\int_{\mathbb{M}}\big(\phi_\xi(x)+1\big)\chi_\xi(x)\varphi_j(x)\mathrm{d}\mu(x)\right)\right].
\end{aligned}$$

The second factor can be bounded by using Corollary A.13, along with the cutoff function ϕ_ξ being bounded by 1 and $\|\varphi_j\|_2 = 1$:

$$\left|\int_{\mathbb{M}}\big(\phi_\xi(x)+1\big)\chi_\xi(x)\varphi_j(x)\mathrm{d}\mu(x)\right| \le 2\|\chi_\xi\|_{L_2(\mathbb{M})} \le 2\Lambda h^m\|\chi_\xi\|_{W_2^m(\mathbb{M})}$$

To bound the first factor, start by using Corollary A.16 and Lemma 5.2 to obtain

$$\begin{aligned}
\left|\int_{\mathfrak{a}}\big(\phi_\xi(x)\big)\chi_\xi(x)\overline{\varphi_j(x)}\mathrm{d}\mu(x)\right| &\le \left(\int_{\mathfrak{a}}|\phi_\xi\chi_\xi|^2\mathrm{d}\mu(x)\right)^{1/2}\\
&\le \Lambda h^m\|\phi_\xi\chi_\xi\|_{W_2^m(\mathfrak{a})} \le \Lambda K' h^m\|\chi_\xi\|_{W_2^m(\mathfrak{a})}.
\end{aligned}$$

Next, from Corollary A.17 we have that

$$\left|\int_{\mathfrak{b}^c}\chi_\xi(x)\overline{\varphi_j(x)}\mathrm{d}\mu(x)\right| \le \left(\int_{\mathfrak{b}^c}|\chi_\xi|^2\mathrm{d}\mu(x)\right)^{1/2} \le \Lambda h^m\|\chi_\xi\|_{W_2^m(\mathfrak{b}^c)}.$$

So the first factor is bounded by

$$\Lambda K' h^m\|\chi_\xi\|_{W_2^m(\mathfrak{a})} + \Lambda h^m\|\chi_\xi\|_{W_2^m(\mathfrak{b}^c)} \le \Lambda(K'+1)h^m\|\chi_\xi\|_{W_2^m(\mathfrak{b}^c)},$$

and the product itself is bounded by

$$C'h^{2m}\|\chi_\xi\|_{W_2^m(\mathfrak{b}^c)}\|\chi_\xi\|_{W_2^m(\mathbb{M})}, \text{ where } C' = 2\Lambda^2(K'+1).$$

Putting these bounds together gives us

$$\sum|Q(\lambda_j)|\left|\left|\int_{\mathbb{M}}\phi_\xi(x)\chi_\xi(x)\overline{\varphi_j(x)}\mathrm{d}\mu(x)\right|^2 - \left|\int_{\mathbb{M}}\chi_\xi(x)\overline{\varphi_j(x)}\mathrm{d}\mu(x)\right|^2\right|$$
$$\leq C'C_Q h^{2m}\|\chi_\xi\|_{W_2^m(\mathbb{M})}\|\chi_\xi\|_{W_2^m(\mathfrak{b}^c)}.$$

Thus for h sufficiently small, say for $C'C_Q h^{2m}\|\chi_\xi\|_{W_2^m(\mathbb{M})} \leq \frac{a_m}{8}\|\chi_\xi\|_{W_2^m(\mathfrak{b}^c)}$, which follows by taking

$$C_0 := \frac{8C'C_Q}{a_m},$$

we have

$$\sum|Q(\lambda_j)|\left|\left|\int_{\mathbb{M}}\phi_\xi(x)\chi_\xi(x)\overline{\varphi_j(x)}\mathrm{d}\mu(x)\right|^2 - \left|\int_{\mathbb{M}}\chi_\xi(x)\overline{\varphi_j(x)}\mathrm{d}\mu(x)\right|^2\right| \leq \frac{a_m}{8}\|\chi_\xi\|^2_{W_2^m(\mathfrak{b}^c)}. \quad (5.7)$$

We note from (4.4) that

$$\frac{a_m}{4}\|\chi_\xi\|^2_{W_2^m(\mathfrak{b}^c)} \leq [\chi_\xi]^2_{k_m,W_2^m(\mathfrak{b}^c)} \quad (5.8)$$

and by subtracting right and left sides of (5.7) from the left and right sides of (5.8) the lemma follows since then

$$\frac{a_m}{8}\|\chi_\xi\|^2_{W_2^m(\mathfrak{b}^c)}$$
$$\leq [\chi_\xi]^2_{k_m,\mathfrak{b}^c} - \left|\sum Q(\lambda_j)\left(\left|\int_{\mathbb{M}}\phi_\xi(x)\chi_\xi(x)\overline{\varphi_j(x)}\mathrm{d}\mu(x)\right|^2 - \left|\int_{\mathbb{M}}\chi_\xi(x)\overline{\varphi_j(x)}\mathrm{d}\mu(x)\right|^2\right)\right|$$
$$\leq [\phi_\xi\chi_\xi]^2_{k_m,\mathfrak{b}^c} = [\phi_\xi\chi_\xi]^2_{k_m,\mathfrak{a}} \leq \overline{a}K'\|\chi_\xi\|^2_{W_2^m(\mathfrak{a})}$$

where the last inequality follows from (4.3). The result follows with $K = 8\overline{a}K'/a_m$. □

We are now ready for the full result.

Theorem 5.5. *Suppose that $\mathbb{M}$ is a compact d-dimensional Riemannian manifold satisfying Assumption 4.1. Suppose further that $m > d/2$ and that k_m satisfies Definition 3.2. There exist positive constants h_0, ν and C, depending only on m, $\mathbb{M}$ and the operator $\mathcal{L}_m$ so that if the set of centers Ξ is quasiuniform with mesh ratio ρ and has density $h \leq H_1$ then the Lagrange functions for interpolation by k_m with auxiliary space $\Pi_{\mathcal{J}}$ satisfy*

$$|\chi_\xi(x)| \leq C\rho^{m-d/2}\max\left(\exp\left(-\frac{\nu}{h}\min\big(d(x,\xi),\mathrm{r}_{\mathbb{M}}\big)\right), h^{2m}\right). \quad (5.9)$$

Furthermore, for any $0 < \epsilon \leq 1$ for which $m > d/2$, there is a constant C depending only on $m, \mathbb{M}, \rho$ and ϵ, so that the Lagrange functions satisfy

$$|\chi_\xi(x) - \chi_\xi(y)| \leq C\left(\frac{d(x,y)}{q}\right)^\epsilon. \quad (5.10)$$

Proof. Let r_0 be the smallest radius r so that $\|\chi_\xi\|_{W_2^m(\mathbf{b}(p,r)^c)} \le C_0 h^{2m} \|\chi_\xi\|_{W_2^m(\mathbb{M})}$. Since $r \mapsto \|\chi_\xi\|_{W_2^m(\mathbf{b}(p,r)^c)}$ is decreasing, $r_0 \le \operatorname{diam}(\mathbb{M})$. Assume without loss that $r_0 \le \mathrm{r}_\mathbb{M}$, since otherwise the proof proceeds exactly as in Theorem 5.3.

Set $t = h/\gamma$, and note that for $t \le r \le r_0$, Lemma 5.4 implies that

$$\|\chi_\xi\|^2_{W_2^m(\mathbf{b}(\xi,r)^c)} \le \epsilon \|\chi_\xi\|^2_{W_2^m(\mathbf{b}(\xi,r-t)^c)}$$

with $\epsilon = (K-1)/K$.

As in the proof of Theorem 5.3

$$\|\chi_\xi\|^2_{W_2^m(\mathbf{b}(\xi,r)^c)} \le \epsilon^{-1} e^{-\nu r/h} \|\chi_\xi\|^2_{W_2^m(\mathbb{M})} \le C e^{-\nu r/h} q^{2m-d}.$$

Where we have set $\nu := -\gamma \log \epsilon$. Since $\epsilon = \frac{K}{K+1} < 1$, it follows that $\nu > 0$. The last inequality follows from (5.6). On the other hand, for $r \ge r_0$, we have that $\|\chi_\xi\|_{W_2^m(\mathbf{b}(p,r)^c)} \le C_0 h^{2m} \|\chi_\xi\|_{W_2^m(\mathbb{M})} \le C C_0 q^{2m-d} h^{2m}$, by (5.6). Therefore,

$$\|\chi_\xi\|_{W_2^m(\mathbf{b}(p,r)^c)} \le C q^{2m-d} \max(h^{2m}, e^{-\nu r/h})$$

Again, estimate (5.9) follows from the observation that $\chi_\xi(x)$ can be estimated by way of the zeros lemma:

$$|\chi_\xi(x)| \le C h^{m-d/2} \|\chi_\xi\|_{W_2^m(\mathbf{b}(x,t)\cap \mathbf{b}(\xi,\operatorname{dist}(x,\xi))^c)} \le C h^{m-d/2} \|\chi_\xi\|_{W_2^m\left(\mathbf{b}\left(\xi, d(x,\xi)\right)^c\right)},$$

for $h < \gamma t$.

Similarly, estimate (5.10) follows from Corollary A.15. □

5.3 Implications for Interpolation and Approximation

At this point, we are able to state three important corollaries to Theorem 5.5 that satisfactorily answer the questions concerning bases and approximation properties of V_X discussed in Section 1. These results were previously obtained in [16, 15] for a class of Sobolev kernels. Here, we get them for a much broader, computationally implementable class of kernels. Our first result is that the Lebesgue constant for interpolation is uniformly bounded.

Theorem 5.6 (Lebesgue Constant)**.** *Let $\mathbb{M}$ be a compact Riemannian manifold of dimension d satisfying Assumption 4.1. Suppose further that $m > d/2$ and that k_m satisfies Definition 3.2. For a quasi-uniform set $\Xi \subset \mathbb{M}$, with mesh ratio $h/q \le \rho$, if $h \le H_0$, then the Lebesgue constant, $L = \sup_{\alpha \in \mathbb{M}} \sum_{\xi\in\Xi} |\chi_\xi(\alpha)|$, associated with k_m and $\mathcal{J}$ is bounded by a constant depending only on m, ρ and $\mathbb{M}$.*

Proof. Fix x. Using Theorem 5.5, we estimate the sum as

$$\sum_{\xi\in\Xi} |\chi_\xi(x)| \le \sum_{\xi\in\Xi} C\rho^{m-d/2} \exp\left(-\nu \frac{\min(\operatorname{dist}(x,\xi), \mathrm{r}_\mathbb{M})}{h}\right) + \sum_{\xi\in\Xi} C\rho^{m-d/2} h^{2m} =: I + II$$

The first sum can be treated exactly as in [16, Theorem 4.6], and is bounded independently of h. The second sum, II, can be estimated using the fact that $\#\Xi \le Cq^{-d}$, with a constant $C = \mu(\mathbb{M})/\alpha(\mathbb{M})$, where $\alpha(\mathbb{M}) := \inf_{x\in\mathbb{M}} \inf_{0<r<\mathrm{r}_{\mathbb{M}}} r^{-d}\mu(\mathbf{b}(x,r))$. Thus

$$II \le C\rho^{m-d/2}q^{-d}h^{2m} \le Ch^{2m-d}\rho^{m+d/2}$$

which is bounded since $2m > d$ (indeed it vanishes as $h \to 0$). $\square$

The next consequence is the L_p stability of the Lagrange basis. To this end, we define

$$S(k_m, \mathcal{J}, \Xi) := \left\{ \sum_{\xi\in\Xi} A_\xi k_m(\cdot,\xi) + p \mid p \in \Pi_{\mathcal{J}} \text{ and } \sum A_\xi q(\xi) = 0 \text{ for all } q \in \Pi_{\mathcal{J}} \right\}$$

and use this notation in lieu of V_X used in the introduction.

Theorem 5.7 (Stability of Lagrange Basis)**.** *Under the assumptions of Theorem 5.5, there exist constants $0 < c_1 < c_2$, depending only on k_m, $\mathcal{J}$, $\mathbb{M}$ and ρ so that*

$$c_1\|A_{p,\cdot}\|_{\ell_p(\Xi)} \le \|s\|_{L_p(\mathbb{M})} \le c_2\|A_{p,\cdot}\|_{\ell_p(\Xi)}.$$

holds for all $s = \sum_{\xi\in\Xi} A_\xi\chi_\xi \in S(k_m,\mathcal{J},\Xi)$, with normalized coefficients $A_{p,\xi} := q^{d/p}A_\xi$.

Proof. When $\mathcal{L}_m$ annihilates $\Pi_{\mathcal{J}}$, this is a direct consequence of the pointwise estimates obtained in Theorem 5.3. We observe that the following three conditions hold:

1. The basis $(\chi_\xi)_{\xi\in\Xi}$ is a Lagrange basis.
2. The basis has decay $|\chi_\xi(x)| \le C\rho^{m-d/2}\exp\left(-\frac{\nu}{h}\min\big(d(x,\xi),\mathrm{r}_{\mathbb{M}}\big)\right)$.
3. The basis has the equicontinuity condition $|\chi_\xi(x) - \chi_\xi(y)| \le C_2\left[\frac{\mathrm{dist}(x,y)}{q}\right]^\epsilon$.

Thus the result [15, Theorem 3.10] applies.

In the general case, the result still holds, despite the fact that item 2 may fail. I.e., the Lagrange functions may decay more slowly than the basis functions considered in [15], and a minor modification is required to apply of [15, Theorem 3.10].

The upper bound $\|s\|_{L_p(\mathbb{M})} \le c_2\|A_{p,\cdot}\|_{\ell_p(\Xi)}$ follows directly from the estimate (5.9). Indeed, the case $p=\infty$, is none other than the Lebesgue constant estimate Theorem 5.6, while the $p=1$ case follows by the uniform bound on

$$\|\chi_\xi\|_1 \le C\rho^{m-d/2}\left(Ch^d + \mathrm{vol}(\mathbb{M})\left(h^{2m} + e^{-\nu\mathrm{r}_{\mathbb{M}}/h}\right)\right) \le C\rho^{m+d/2}q^d.$$

The case $1 < p < \infty$ follows by interpolation.

To handle the lower bound, we utilize functions ϕ_ξ, defined in a similar way as in (5.2), satisfying $\phi_\xi \circ \mathrm{Exp}_\xi = \sigma$, with

$$\sigma(x) = \begin{cases} 1 & |x| \le r_0 \\ h^{-2m}e^{-\frac{\nu|x|}{h}} & r_0 < |x| \le \mathrm{r}_{\mathbb{M}} \\ h^{-2m}e^{-\frac{\nu\mathrm{r}_{\mathbb{M}}}{h}} & |x| > \mathrm{r}_{\mathbb{M}} \end{cases} \quad \text{or} \quad \sigma(x) = \begin{cases} 1 & |x| \le r_0 \\ h^{-2m}e^{-\frac{\nu|x|}{h}} & r_0 < |x| \end{cases} \tag{5.11}$$

and with threshold value $r_0 := -\frac{2m}{\nu} h \log h$ (the second definition is chosen if $\mathrm{r}_{\mathbb{M}} < r_0$). It follows that

$$\chi_\xi = \chi_\xi \phi_\xi + \chi_\xi(1-\phi_\xi) =: g_\xi + b_\xi,$$

and g_ξ satisfies items 1–3 above (in particular, item 3 follows since ϕ_ξ is bounded and Lip(1), with Lipschitz constant ν/h) and [15, Theorem 3.10] applies. In particular, there is $c_1 > 0$ so that $\|\sum_{\xi\in\Xi} A_\xi g_\xi\|_p \geq c_1 \|A_{p,\cdot}\|_{\ell_p(\Xi)}$.

On the other hand, $|b_\xi(x)| \leq h^{2m}$, implies that $\|\sum_{\xi\in\Xi} A_\xi b_\xi\|_p \leq C\rho^d \|A_{p,\cdot}\|_{\ell_p \Xi} h^{2m-d}$ since

$$\int_{\mathbb{M}} |\sum_{\xi\in\Xi} A_\xi b_\xi(x)| \mathrm{d}x \leq \|A\|_{\ell_1(\Xi)} \mu(\mathbb{M}) h^{2m} \quad \text{and} \quad \max_{x\in\mathbb{M}} |\sum_{\xi\in\Xi} A_\xi b_\xi(x)| \leq C\rho^d \|A\|_{\ell_\infty(\Xi)} h^{2m-d}.$$

Thus, for $s = \sum_{\xi\in\Xi} A_\xi \chi_\xi$,

$$\|s\|_p = \|\sum_{\xi\in\Xi} A_\xi \chi_\xi\|_p \geq \left(2^{1-p} \|\sum_{\xi\in\Xi} A_\xi g_\xi\|_p^p - \|\sum_{\xi\in\Xi} A_\xi b_\xi\|_p^p\right)^{1/p} \geq \left(\frac{c_1}{2} - o(h)\right) \|A_{p,\cdot}\|_{\ell_p(\Xi)},$$

where we have used the inequality $|\sum_{\xi\in\Xi} A_\xi g_\xi|^p \leq 2^{p-1}\left(|\sum_{\xi\in\Xi} A_\xi \chi_\xi|^p + |-\sum_{\xi\in\Xi} A_\xi b_\xi|^p\right)$. $\square$

Our final consequence treats the L_p stability of the L_2 projector. This was a primary goal of [15], and, in light of Theorem 5.5, we can produce a similar result here, with a minor modification to handle the slower decay of the Lagrange functions.

Let $V : \mathbb{C}^\Xi \to S(k_m, \mathcal{J}, \Xi)$ be a basis "synthesis operator" $V : (A_\xi)_{\xi\in\Xi} \mapsto \sum_{\xi\in\Xi} A_\xi v_\xi$, for a basis $(v_\xi)_{\xi\in\Xi}$ of $S(k_m, \mathcal{J}, \Xi)$. Likewise, let $V^* : L_1(\mathbb{M}) \to \mathbb{C}^\Xi$ be its formal adjoint $V^* : f \mapsto \left(\langle f, v_\xi\rangle\right)_{\xi\in\Xi}$. The L_2 projector is then $T_\Xi = V(V^*V)^{-1}V^* : L_1(\mathbb{M}) \to S(k_m, \mathcal{J}, \Xi)$, in the sense that when $f \in L_2(\mathbb{M})$, $T_\Xi f$ is the best L_2 approximant to f from $S(k_m, \mathcal{J}, \Xi)$.

The L_2 norm of this projector is 1 (it being an orthogonal projector), while the L_p and L_p' norms are equal, because it is self-adjoint. Thus, to estimate its L_p operator norm ($1 \leq p \leq \infty$), it suffices to estimate its L_∞ norm.

Theorem 5.8. *Under the assumptions of Theorem 5.5, for all $1 \leq p \leq \infty$, the L_p operator norm of the L_2 projector T_Ξ is bounded by a constant depending only on $\mathbb{M}$, ρ k_m and $\mathcal{J}$.*

Proof. When $\mathcal{L}_m$ annihilates $\Pi_{\mathcal{J}}$, Theorem 5.3 and Theorem 5.7 satisfy the conditions of [15, Theorem 5.1] (the Lagrange basis is stable and rapidly decaying), and the result follows.

In the general case, we cannot directly apply this theorem, because the basis does not decay rapidly enough. We take as our basis $v_\xi = \chi_{\xi,2} := q^{-d/2}\chi_\xi$, the L_2 normalized Lagrange basis. It follows from Theorem 5.7 that $\|V\|_{\ell_\infty(\Xi)\to L_\infty(\mathbb{M})} \leq c_2 q^{-d/2}$ and $\|V^*\|_{L_\infty(\mathbb{M})\to\ell_\infty(\Xi)} \leq c_2 q^{d/2}$. Thus, to estimate the L_∞ operator norm of T_Ξ (and thereby all other L_p norms), it suffices to estimate the $\ell_\infty(\Xi) \to \ell_\infty(\Xi)$ norm of the inverse Gram matrix $(V^*V)^{-1}$.

We make the split $g_\xi = \chi_\xi\phi_\xi$ and $b_\xi = \chi_\xi(1-\phi_\xi)$ with $\phi_\xi \circ \mathrm{Exp}_\xi = \sigma$ defined as in (5.11). And note that $\chi_{\xi,2} = \chi_{\xi,2}\phi_\xi + \chi_{\xi,2}(1-\phi_\xi) =: g_{\xi,2} + b_{\xi,2}$,

It follows that $V^*V = G + B$, with $G_{\xi,\zeta} = \langle g_{\xi,2}, g_{\zeta,2}\rangle_{L_2(\mathbb{M})}$.

The functions (g_ξ) are a Lagrange basis, in the sense that $g_\xi(\zeta) = \delta_{\xi,\zeta}$ (although they span a different space than $S(k_m, \mathcal{J}, \Xi)$ and, as observed in the proof of Theorem 5.7, they are L_p stable. They also satisfy the decay conditions of [15, Proposition 4.1], and by applying this result we see that $\|G^{-1}\|_\infty$ is bounded by a constant.

On the other hand, $|B_{\xi,\zeta}| \le |\langle g_{\xi,2}, b_{\zeta,2}\rangle| + |\langle b_{\xi,2}, g_{\zeta,2}\rangle| + |\langle b_{\xi,2}, b_{\zeta,2}\rangle| \le Ch^{2m}$, and $\|B\|_\infty \le \max_\xi \sum_\zeta |B_{\xi,\zeta}| \le C\rho^d h^{2m-d}$. The theorem follows by noting that $(V^*V) = G(\mathrm{Id} + G^{-1}B)$, and, hence, $\|(V^*V)^{-1}\|_\infty \le \|G^{-1}\|_\infty (1 + o(h))$.

□

5.4 Spheres and $SO(3)$

We now explore some further consequences of the results of the previous section. We will shortly see that Theorems 5.6 and 5.8 imply that I_Ξ and T_Ξ are *near-best*. In some important cases, we can then use these projectors to observe precise rates of convergence for interpolation and least squares minimization, better rates than were previously known.

For the kernels considered in section 3.2, theoretical approximation results are known in some special cases, including spheres and $SO(3)$. The difficulty is that these results are often not practical, because they are derived from approximation schemes that are difficult to implement. The good news is that the stability of the schemes I_Ξ and T_Ξ imply that these operators, which are associated with *practical* schemes, inherit the *same* convergence rates. Indeed, for a normed linear space Y and a bounded projector $P : Y \to Y$, one has for $f \in Y$,

$$\|f - Pf\| = \inf_{s \in \mathrm{ran}P} \|f - s + Ps - Pf\| \le (1 + \|P\|)\mathrm{dist}(f, \mathrm{ran}P). \tag{5.12}$$

This fundamental observation is known as a Lebesgue inequality, and we employ it with $P = I_\Xi$ and $Y = C(\mathbb{M})$ as well as with $P = T_\Xi$ and $Y = L_p(\mathbb{M})$. In recent years, a concerted effort has been undertaken[4] to understand the general L_p convergence rates (i.e., the behavior of $\mathrm{dist}(f, S(k_m, \mathcal{J}, \Xi))_p$ as Ξ becomes dense in $\mathbb{M}$) of certain well-known kernels in terms of smoothness assumptions on the target function f and on the density of the point-set Ξ, measured by the fill distance h.

To measure smoothness of the target function, we make use of the classical (Sobolev, Besov) smoothness spaces introduced in Definition 2.1 and Definition 2.3, with the exception that for approximation in L_∞, we make the (usual) replacement of C^{2m} for W_∞^{2m} (but using the same norm). As a shorthand, we capture the smoothness spaces we use by means of a common notation, $\mathcal{W}_p^s(\mathbb{M})$. For $m > d/2$ denote the space $\mathcal{W}_p^s(\mathbb{M})$ by

- $\mathcal{W}_p^s(\mathbb{M}) = C^{2m}(\mathbb{M})$, when $p = \infty$ and $s = 2m$
- $\mathcal{W}_p^s(\mathbb{M}) = W_p^{2m}(\mathbb{M})$ when $1 \le p < \infty$ and $s = 2m$

[4] This stands in contrast to the more classical, mainstream theory of kernel approximation, where approximation properties are investigated and understood only for functions coming from the reproducing kernel (semi-)Hilbert space associated with a conditionally positive definite kernel. Obtaining an understanding outside of this context has generally required indirect, theoretical approximation schemes, and it has not been obvious, until now, that such results would have practical consequences.

- $\mathcal{W}_p^s(\mathbb{M}) = B_{p,\infty}^s(\mathbb{M})$ when $1 \le p \le \infty$ and $0 < s < 2m$.

Corollary 5.9. *For $\mathbb{M} = \mathbb{S}^d$, and $m > d/2$ the surface splines introduced in Example 3: $k_m(x,y) = \phi_s(x\cdot, y)$ satisfy the following. There is a constant C so that, for a sufficiently dense set $\Xi \subset \mathbb{S}^d$, and for $f \in \mathcal{W}_p^s$.*

1. *For $f \in \mathcal{W}_\infty^s$, $\|I_\Xi f - f\| \le C h^s \|f\|_{\mathcal{W}_\infty^s}$*
2. *For $1 \le p \le \infty$ and for $f \in \mathcal{W}_p^s$, $\|T_\Xi f - f\|_p \le h^s \|f\|_{\mathcal{W}_\infty^s}$.*

Proof. As with the Sobolev kernels, ϕ_s is of the form $G_\beta + \psi * G_\beta$ as considered in [22], the result follows direct from [22, Theorem 6.8]. Alternatively, it follows from [14, Theorem 6.1], which treats kernels on the sphere of the type in Definition 3.2. The Besov space result follows from [22, Corollary 6.13] or [14, Corollary 6.2]. □

Corollary 5.10. *For $\mathbb{M} = SO(3)$, and $m \ge 2$ The surface splines, $\mathbf{k}_m$, introduced in Example 4 satisfy the following. There is a constant C so that, for a sufficiently dense set $\Xi \subset SO(3)$, and for $f \in \mathcal{W}_p^s$.*

1. *For $f \in \mathcal{W}_\infty^s$, $\|I_\Xi f - f\| \le C h^s \|f\|_{\mathcal{W}_\infty^s}$*
2. *For $1 \le p \le \infty$ and for $f \in \mathcal{W}_p^s$, $\|T_\Xi f - f\|_p \le h^s \|f\|_{\mathcal{W}_\infty^s}$.*

Proof. This follows from [17, Theorem 9] for the case of full smoothness and from [17, Theorem 12] when $0 < s < 2m$. □

A A Zeros Lemma for Lipschitz Domains on Manifolds

Results concerning Sobolev bounds on functions with many zeros are known for Lipschitz domains in $\mathbb{R}^d$ [25, 26]. Our aim is to extend these results to certain Lipschitz domains on manifolds. Before we do that, however, we will need to improve the $\mathbb{R}^d$ results in [25, 26].

A.1 Lipschitz domains in $\mathbb{R}^d$

Consider a domain $\Omega \subset \mathbb{R}^d$ that is bounded, has a Lipschitz boundary, and satisfies an interior cone condition, where the cone C_Ω has a maximum radius R_0 and aperture[5] 2φ. Of course, the cone condition will be obeyed if we use any radius $0 < R \le R_0$. The theorem that we will give below requires covering Ω with certain star-shaped domains.

We will say that a domain $\mathcal{D}$ is *star shaped with respect to a ball* $B(x_c, r) := \{x \in \mathbb{R}^d : |x - x_c| < r\}$ if, for every $x \in \mathcal{D}$, the closed convex hull of $\{x\} \cup B$ is contained in $\mathcal{D}$ [4, Chapter 4]. For $\mathcal{D}$ bounded, there is a measure of how close to spherical $\mathcal{D}$ is; namely, the *chunkiness parameter* γ [4, Definition 4.2.16]. This is defined as the ratio of $d_\mathcal{D}$ to the radius of the largest ball relative to which $\mathcal{D}$ is star shaped. When $\mathcal{D}$ is a sphere, $\gamma = 2$. If there is a ball $B(x_c, R) \supseteq \mathcal{D}$, then $r < d_\mathcal{D} < 2R$ and $\gamma \le \frac{2R}{r}$. Finally, such domains satisfy an interior cone condition and certain Sobolev bounds, which are stated in the next two propositions.

[5]Aperture here is the angle across the cone, 2φ in this case. In optics, aperture would be φ.

Proposition A.1 ([25, Proposition 2.1])**.** *If $\mathcal{D}$ is bounded, star shaped with respect to $B(x_c, r)$ and contained in $B(x_c, R)$, then every $x \in \mathcal{D}$ is the vertex of a cone $C_{\mathcal{D}} \subset \mathcal{D}$ having radius r and aperture $\theta := 2\arcsin\left(\frac{r}{2R}\right)$.*

This proposition also implies that the chunkiness parameter for $\mathcal{D}$ is bounded in terms of the aperture:

$$\gamma \leq \frac{2R}{r} = \csc(\theta/2).$$

Proposition A.2 ([16, Proposition 3.5])**.** *Let $\mathcal{D} \subset \mathbb{R}^d$ be as above, $m \in \mathbb{N}$ and $\mathsf{p} \in \mathbb{R}$, $1 \leq \mathsf{p} \leq \infty$. Assume $m > d/\mathsf{p}$ when $\mathsf{p} > 1$, and $m \geq d$, for $\mathsf{p} = 1$. If $u \in W_{\mathsf{p}}^m(\mathcal{D})$ satisfies $u|_X = 0$, where $X = \{x_1, \ldots, x_N\} \subset \mathcal{D}$ and if $h = h_X \leq \frac{d_{\mathcal{D}}}{16m^2\gamma^2}$, then*

$$|u|_{W_{\mathsf{p}}^k(\mathcal{D})} \leq C_{m,d,\mathsf{p}} \gamma^{d+2k} d_{\mathcal{D}}^{m-k} |u|_{W_{\mathsf{p}}^m(\mathcal{D})} \tag{A.1}$$

$$\|u\|_{L_\infty(\mathcal{D})} \leq C_{m,d,\mathsf{p}} \gamma^d d_{\mathcal{D}}^{m-d/\mathsf{p}} |u|_{W_{\mathsf{p}}^m(\mathcal{D})}. \tag{A.2}$$

Our next task is to obtain Sobolev bounds for the domain $\Omega \subset \mathbb{R}^d$ that are similar those in (A.1). The idea is to cover Ω with star-shaped domains. To do that, we will use a construction due to Duchon [8]. With R_0, 2φ being the radius and aperture for the cone C_Ω, and $0 < R \leq R_0$, let

$$r := 2RF(\varphi), \text{ where } F(\varphi) := \frac{\sin(\varphi)}{4(1+\sin(\varphi))}, \text{ and } T_r := \left\{ t \in \frac{2r}{\sqrt{d}}\mathbb{Z}^d \colon B(t,r) \subset \Omega \right\}. \tag{A.3}$$

For $t \in T_r$, let $\mathcal{D}_t$ be the set of all $x \in \Omega$ such that the closed convex hull of $\{x\} \cup B(t,r)$ is contained in $\Omega \cap B(t,R)$. From [25, Lemma 2.11], we have that each $\mathcal{D}_t$ is star shaped with respect to the ball $B(t,r)$, and satisfies $B(t,r) \subseteq \mathcal{D}_t \subseteq \Omega \cap B(t,R)$, $d_{\mathcal{D}_t} < 2R$. Because $2R/r = 1/F(\varphi)$, the aperture for $C_{\mathcal{D}_t}$ is

$$\theta = 2\arcsin(1/F(\varphi)),$$

and the chunkiness parameter γ_t for $\mathcal{D}_t$ is uniformly bounded:

$$2 \leq \gamma_t < \frac{2R}{r} = \frac{1}{F(\varphi)}. \tag{A.4}$$

We also have that $\Omega = \bigcup_{t \in T_r} \mathcal{D}_t$, that $\#T_r < C_d \operatorname{vol}(\Omega)(F(\varphi)R)^{-d}$, and that $\operatorname{vol}(\mathcal{D}_t) \leq C_d R^d$.

The integer-valued simple function $\sum_{t \in T_r} \chi_{B(t,R)}(x)$ is the number of $B(t,R)$'s that contain x. This is easily bounded above by $M_{d,\varphi}$, maximum number of such balls intersecting a fixed one, say $B(0,R)$. A little geometry shows that

$$M(d,\varphi) \leq (2R/r+1)^d \leq 2^d/(F(\varphi)^d$$

Note that the existence of $M_{d,\varphi}$ implies that for any function f in $L_1(\Omega)$ we have

$$\sum_t \int_{\mathcal{D}_t} |f(x)|dx = \int_\Omega \sum_t \chi_{\mathcal{D}_t}(x)|f(x)|dx \leq M_{d,\varphi} \int_\Omega |f(x)|dx \leq (2^d/(F(\varphi)^d) \int_\Omega |f(x)|dx. \tag{A.5}$$

Lemma A.3. *Suppose that $h = h_{X,\Omega}$ satisfies $h \le \frac{R}{8m^2}F(\varphi)^2$, then (A.1) and (A.2) hold uniformly in t for $\mathcal{D}_t$, provided γ_t and $d_{\mathcal{D}_t}$ are replaced by $1/F(\varphi)$ and $2R$, respectively.*

Proof. The mesh norm for Ω satisfies

$$h \le m^{-2}(\underbrace{2RF(\varphi))}_{r} \times (F(\varphi)/16) < r,$$

since $F(\varphi) < 1$. It follows that $B(x_c, r) \cap X \ne \emptyset$, and so $\mathcal{D}_t \cap X$ contains at least one point of X. From this we have that $h_{\mathcal{D}_t \cap X} \le h$. The lemma then follows from the bound on h being less than the one required in Proposition A.2. $\square$

We wish to prove the following result, which differs from an earlier result in [25, Theorem 2.12] in that it applies to cases in which the index $k \le m-1$, as opposed to $k < m - n/p$.

Theorem A.4 (Euclidean Case). *Suppose that Ω is a Lipschitz domain obeying a cone condition, where the cone C_Ω has radius R_0 and aperture 2φ. Let k, m, and p be as in Proposition A.2, and and let $X \subset \Omega$ be a discrete set with mesh norm h satisfying*

$$h < \frac{R_0}{8m^2}F(\varphi)^2. \tag{A.6}$$

If $u \in W_p^m(\Omega)$ satisfies $u|_X = 0$, then

$$|u|_{W_\mathsf{p}^k(\Omega)} \le \frac{2^{d/\mathsf{p}}(4m)^{2m-2k}C_{m,d,\mathsf{p}}}{F(\varphi)^{2m+d+d/\mathsf{p}}} h^{m-k} |u|_{W_\mathsf{p}^m(\Omega)} \tag{A.7}$$

and

$$\|u\|_{L_\infty(\Omega)} \le \frac{(4m)^{2m-2d/\mathsf{p}}C_{m,d,\mathsf{p}}}{F(\varphi)^{2m+d-2d/\mathsf{p}}} h^{m-d/\mathsf{p}} |u|_{W_\mathsf{p}^m(\Omega)}. \tag{A.8}$$

Proof. Given h, choose $R = 8m^2h/F(\varphi)^2 < R_0$. Applying Lemma A.3 and Proposition A.2 to the domain $\mathcal{D}_t$ then results in the bound

$$|u|_{W_\mathsf{p}^k(\mathcal{D}_t)} \le \frac{(4m)^{2m-2k}C_{m,d,\mathsf{p}}}{F(\varphi)^{d+2m}} h^{m-k} |u|_{W_\mathsf{p}^m(\mathcal{D}_t)}$$

We will follow the proof in [25, Theorem 2.12]. Summing over t on both sides of the previous inequality, using $\Omega = \cup_t \mathcal{D}_t$ and applying (A.5), we have that

$$|u|^\mathsf{p}_{W_\mathsf{p}^k(\Omega)} \le \left(\frac{(4k)^{2m-2k}C_{m,d,\mathsf{p}}h^{m-k}}{F(\varphi)^{d+2m}} \right)^\mathsf{p} (2^d/(F(\varphi)^d)|u|^\mathsf{p}_{W_\mathsf{p}^m(\Omega)},$$

from which (A.7) is immediate. The bound on $\|u\|_{L_\infty(\Omega)}$ follows similarly. $\square$

A.2 Lipschitz domains in $\mathbb{M}$

A domain Ω on a smooth, compact Riemannian manifold $\mathbb{M}$ satisfies an interior cone condition if there is a cone $C \subset \mathbb{R}^n$ with center 0, aperture 2φ, and radius R such that, with some orientation of C, $\mathrm{Exp}_p : C \to C_p \subset \Omega$. That is, the image of the fixed cone C is a geodesic cone C_p contained in Ω. In addition, Ω satisfies the *uniform cone condition* if, for every $p_0 \in \partial\Omega$ and some orientation of C, $\mathrm{Exp}_p(C \setminus \{0\}) \subseteq \Omega$ for all $p \in \mathbf{b}(p_0, r) \cap \overline{\Omega}$. Finally, Ω is said to be *locally strongly Lipschitz* [23, 19] if for every $p_0 \in \partial\Omega$ there is a local chart (U, ψ), $\psi : U \to \mathbb{R}^n$,with $\psi(p_0) = 0$, a Lipschitz function $\lambda : \mathbb{R}^{n-1} \to \mathbb{R}$, with $\lambda(0) = 0$, and an $\varepsilon > 0$ such that

$$\psi(U \cap \Omega) = \{(x', \lambda(x') + t) : 0 < t < \varepsilon, x' \in \mathbb{R}^{n-1}, |x'| < \varepsilon\}.$$

Our approach to a manifold analogue of Theorem A.4 is to employ a set of points for $\mathbb{M}$ that are similar to those described in (A.3). The set that we need is described and studied in [13, §3]. Let $\varepsilon > 0$. There exists an ordered set of points $\{p_1, \dots, p_N\} \subset \mathbb{M}$ such that the $\cup_{j=1}^N \mathbf{b}(p_j, \varepsilon) = \mathbb{M}$ and such that the balls $\mathbf{b}(p_j, \varepsilon/2)$ are disjoint. Such a set is called a *minimal ε-net* in $\mathbb{M}$.[6] It has the following two important properties: First, there is a number $N_1 = N_1(\varepsilon, \mathbb{M})$ for which $N \le N_1$. Second, there exists an integer $N_2 = N_2(\mathbb{M}) \ge 1$ such that for any $p \in \mathbb{M}$ the ball $\mathbf{b}(p, \varepsilon)$ *intersects at most* N_2 of the balls $\mathbf{b}(p_j, \varepsilon)$. It is remarkable that N_2 is *independent* of ε and, in fact, depends only on general properties of $\mathbb{M}$ itself. We will need a slightly stronger version of this result.

Lemma A.5. *Let $\{p_1, \dots, p_N\}$ be a minimal ε-set, $p \in \mathbb{M}$, and let $1 \le \alpha$. Suppose $\varepsilon \le d_{\mathbb{M}}/\alpha$, where $d_{\mathbb{M}}$ is the diameter of $\mathbb{M}$. Then the cardinality $s := \#\{p_j \colon \mathbf{b}(p, \alpha\varepsilon) \cap \mathbf{b}(p_j, \varepsilon) \neq \emptyset\} \le (4\alpha + 1)^d e^{\frac{3(d-1)}{\sqrt{|\kappa|}} d_{\mathbb{M}}}$.*

Proof. The argument used in [13, Lemma 3.3] gives, *mutatis mutandis*,

$$s \le \frac{\int_0^{(2\alpha+\frac{1}{2})\varepsilon} \sinh^{d-1}(\sqrt{|\kappa|}\, t)dt}{\int_0^{\varepsilon/2} \sinh^{d-1}(\sqrt{|\kappa|} t)dt} =: H(\alpha, \varepsilon, \kappa) = H(\alpha, \varepsilon/\sqrt{|\kappa|}, 1)$$

where $(d-1)\kappa$ is a lower bound on the Ricci curvature of $\mathbb{M}$. Making use of $1 \le \sinh(x)/x \le e^x$, we see that

$$d^{-1}x^d \le \int_0^x t^{d-1}dt \le \int_0^x \sinh^{d-1}(t)dt \le d^{-1}x^d e^{(d-1)x},$$

and consequently that

$$H(\alpha, \varepsilon, \kappa) \le (4\alpha + 1)^d e^{(d-1)(2\alpha+\frac{1}{2})\varepsilon/\sqrt{|\kappa|}} \le (4\alpha + 1)^d e^{\frac{3(d-1)}{\sqrt{|\kappa|}} d_{\mathbb{M}}},$$

which completes the proof. ∎

[6] An ε-net is a set of points $X = \{p_1, \dots, p_N\}$ for which $\bigcup \mathbf{b}(p_j, \varepsilon)$ covers $\mathbb{M}$ – in other words, for which $h(X, \mathbb{M}) \le \varepsilon$. Likewise, a *minimal ε-net* is quasiuniform, with separation distance $q \ge \epsilon/2$ and mesh ratio $h/q \le 2$.

Lemma A.6. *Let $R < \mathrm{r}_{\mathbb{M}}/3$, $\varphi \in (0, \pi/2]$ and $\varepsilon = \frac{\Gamma_1 R \sin(\varphi)}{2(1+\sin(\varphi))}$. If $\{p_1, \dots, p_N\}$ is an ε-set and if C_p is a geodesic cone with center p, radius R, and angle φ, then for some j we have that $\mathbf{b}(p_j, \varepsilon) \subset C_p$.*

Proof. We will work in normal coordinates on T_pM, where the cone C has vertex at the origin and $e_n = (0, \dots, 1)$ is chosen to be along the axis of C. The largest Euclidean ball in C has radius $\rho = R\sin(\varphi)/(1+\sin(\varphi))$ and center $x_c = (R-\rho)e_n$. It follows that any ball having its center a Euclidean distance $\rho/2$ from x_c and having its radius less than $\rho/2$ is also contained in C. Let $p_c = \mathrm{Exp}(x_c)$. Since the balls $\mathbf{b}(p_j, \varepsilon)$, $j = 1, \dots, N$, cover $\mathbb{M}$, we can find p_j such that $p_c \in \mathbf{b}(p_j, \varepsilon)$.

Let $x_j = \mathrm{Exp}_p^{-1}(p_j)$. Equation (2.3) implies that $|x_c - x_j| \le \mathrm{dist}(p_c, p_j)/\Gamma_1 < \varepsilon/\Gamma_1 = \rho/2$. Now consider the ball $\mathbf{b}(p_j, \Gamma_1\rho/2)$. Let $q \in \mathbf{b}(p_j, \Gamma_1\rho/2)$ and let $x = \mathrm{Exp}_p^{-1}(q)$. Applying equation (2.3) then yields that $|x - x_j| \le \mathrm{dist}(p, q)/\Gamma_1 < \rho/2$, and consequently that $\mathbf{b}(p_j, r) \subset C_p$, with $r \le \rho/2 = \frac{\Gamma_1 R \sin(\varphi)}{2(1+sin(\varphi))}$. ☐

Our goal is now to cover Ω with domains analogous to those used in the previous section. To that end, let $R \le R_\Omega$, fix

$$r := \frac{\Gamma_1 R \sin(\varphi)}{2(1+\sin(\varphi))} = 2F(\varphi)R \tag{A.9}$$

and find a minimal ε-net (with $\varepsilon = r$) $\{p_1, \dots, p_N\}$ and set $T_r := \{p_j \colon \mathbf{b}(p_j, r) \subset \Omega\}$. Because Ω obeys a uniform cone condition, with radius R_Ω and angle φ, Lemma A.6 implies that T_r is nonempty.

Next, for each $p_j \in T_r$, let $\mathcal{D}_j$ be the set of all $p \in \Omega \cap \mathbf{b}(p_j, R)$ such that the geodesic convex hull of $\{p\} \cup \mathbf{b}(p_j, r)$ – i.e., the set comprising all points on every geodesic connecting p to a point in $\mathbf{b}(p_j, r)$ – is contained in $\Omega \cap \mathbf{b}(p_j, R)$. Again by Lemma A.6, for every $p \in \Omega$ there is a $p_j \in T_r$ such that the geodesic cone C_p contains $\mathbf{b}(p_j, r)$. Since this cone also contains the geodesic convex hull of $\{p\} \cup \mathbf{b}(p_j, r)$, it follows that $p \in \mathcal{D}_j$ and, hence, that $\Omega = \cup_{p_j \in T_r} \mathcal{D}_j$.

We claim that the domain $D_j := \mathrm{Exp}_{p_j}^{-1}(\mathcal{D}_j)$ is star shaped with respect to the Euclidean ball $\mathbf{b}(\mathrm{Exp}_{p_j}^{-1}(p_j), r/\Gamma_2)$. To show this, we will need the following lemma.

Lemma A.7. *Let $p \in \mathbb{M}$, $\mathbf{u}, \mathbf{v} \in T_pM$ satisfy $|\mathbf{u}|_p = |\mathbf{v}|_p = 1$, $\alpha := \arccos(\langle \mathbf{u}, \mathbf{v}\rangle) \in (0, \pi]$. If $p_\rho = \mathrm{Exp}_p(\rho\mathbf{u})$, so that $\rho = \mathrm{dist}(p, p_\rho) < \mathrm{r}_{\mathbb{M}}/3$, then the geodesic distance r from p_ρ to the ray along $\mathbf{v}$ satisfies*

$$\Gamma_1 \rho \sin\bigl(\min(\alpha, \pi/2)\bigr) \le r \le \Gamma_2 \rho \sin\bigl(\min(\alpha, \pi/2)\bigr).$$

Proof. Consider the sector in $\mathrm{span}(\mathbf{u}, \mathbf{v})$ formed by $t\mathbf{u} + s\mathbf{v}$, where $s, t \ge 0$. We will work in normal coordinates based at p. The minimum geodesic distance r from p_ρ to geodesic $\mathrm{Exp}_p(s\mathbf{v})$ occurs at a point $\mathrm{Exp}_p(t\mathbf{v})$. In addition, the minimum Euclidean distance r' from $\rho\mathbf{u} = \mathrm{Exp}_p^{-1}(p_\rho)$ to the ray will occur at another point, $t'\mathbf{v}$, where $\mathbf{v}$ is perpendicular to $t'\mathbf{v} - \rho\mathbf{u}$, in the Euclidean sense. These facts imply that $r = \mathrm{dist}(p_\rho, t\mathbf{v}) \le \mathrm{dist}(p_\rho, \mathrm{Exp}_p(t'\mathbf{v})) \le \Gamma_2 |\rho\mathbf{u} - t'\mathbf{v}|_{\mathrm{eucl}}$. Using a little trigonometry, together with $t'\mathbf{v} - \rho\mathbf{u}$ and $\mathbf{v}$ being perpendicular,

we see that $|\rho\mathbf{u} - t'\mathbf{v}|_{\text{eucl}} = \rho\sin(\alpha)$ when $\alpha < \pi/2$, and that $|\rho\mathbf{u} - t'\mathbf{v}|_{\text{eucl}} = |\rho\mathbf{u}|_{\text{eucl}} = \rho$ when $\alpha \geq \pi/2$. In a similar way, we have $r = \text{dist}(p, \text{Exp}_p(t\mathbf{v})) \geq \Gamma_1|\rho\mathbf{u} - t\mathbf{v}|_{\text{eucl}} \geq \Gamma_1|\rho\mathbf{u} - t'\mathbf{v}|_{\text{eucl}} = \Gamma_1\rho\sin\big(\min(\alpha, \pi/2)\big)$. Combining the inequalities completes the proof. □

There is corollary to the lemma that will be useful for smooth surfaces, in particular balls and annuli. We state and prove it now, although it will only become useful after the zeros result Theorem A.11.

Corollary A.8. *Let $q \in \partial\Omega$ and suppose there is a ball $\mathbf{b}(p, \rho)$, $\rho < \mathrm{r}_{\mathbb{M}}/3$, such that $\mathbf{b}(p, \rho) \subset \Omega$ and that $d(q, p) = \rho$. Then, the geodesic cone C_q, with vertex q, axis along the geodesic joining q to p, radius $\Gamma_1\rho/2$ and angle $\varphi = \arcsin(\frac{1}{2\Gamma_2})$ satisfies $C_q \setminus \{q\} \subset \Omega$.*

Proof. Let q' be a point on the lateral side of the cone that is ρ away from p – denote it in coordinates around q by $q' = s\mathbf{v}$. With this, we identify two triangles.

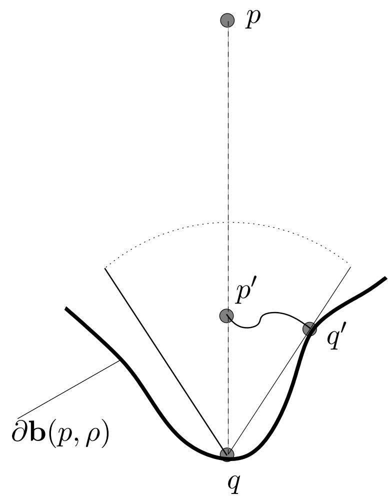

Figure 1: In this figure, p is the center of the ball of radius ρ, q is a point on the boundary, q' is a point simultaneously on the boundary of the ball and on the side of the cone and p' is the nearest point on the ray $\text{Exp}_q(t\mathbf{v})$ to q'.

The first triangle has corners $p = \text{Exp}_q(\rho\mathbf{u})$, $q' = \text{Exp}_q(s\mathbf{v})$ and a point on the ray $\text{Exp}_q(t\mathbf{u})$ with $0 < t < \rho$ (Lemma A.7 guarantees that the vertex at p is acute, since $s \leq \Gamma_1\rho/2$). Let us denote the third corner of this triangle by $p' = \text{Exp}_q(t'\mathbf{u})$ Note that p' is a distance of t' from q and a distance of $\rho - t'$ from p. The triangle inequality gives us that

$$\rho \leq \text{dist}(p', q') + (\rho - t') \quad \longrightarrow \quad t' \leq \text{dist}(p', q').$$

The second triangle we consider has corners q, q' and p', and Lemma A.7 ensures that $\text{dist}(q', p') \leq \Gamma_2 s \sin\alpha = s/2$. So the triangle inequality here gives us that $s \leq \text{dist}(p', q') + t' \leq s/2 + t'$ – so $s/2 \leq t'$.

Combining estimates from both triangles, we see that $s/2 \leq t' \leq \text{dist}(p', q') \leq s/2$, so both t' and $\text{dist}(p', q')$ are $s/2$.

In other words, the curve from p to q' has the same length as the curve from p to p' to q'. By [7, Corollary 3.9, p. 73], any piecewise differentiable curve joining two points – p to p' to q' our case – with length less than or equal to any other such curve is a geodesic. Because this occurs inside $\mathbf{b}(p, \mathrm{r}_{\mathbb{M}})$, where geodesics do not cross, there can only be one geodesic joining p and q'. Since p to p' has to be on the geodesic joining p' to q', and since the length is ρ, q' and q coincide. $\square$

Proposition A.9. *The domain $D_j := \mathrm{Exp}_{p_j}^{-1}(\mathcal{D}_j)$ is star shaped with respect to the Euclidean ball $B(\mathrm{Exp}_{p_j}^{-1}(p_j), \Gamma_1 r/\Gamma_2^2)$. Also, the chunkiness parameter and diameter for D_j satisfy*

$$\gamma_{D_j} \leq \frac{2\Gamma_2^2 R}{\Gamma_1 r} = \left(\frac{\Gamma_2}{\Gamma_1}\right)^2 \frac{4(1+\sin(\varphi))}{\Gamma_1 \sin(\varphi)} = \frac{\Gamma_2^2}{\Gamma_1^2 F(\varphi)} \text{ and } d_{D_j} \leq 2R. \tag{A.10}$$

Proof. We begin by fixing a point $p \in \mathcal{D}_j$. The geodesic convex hull of $\{p\} \cup \mathbf{b}(p_j, r)$ contains a largest cone with vertex p and central axis the geodesic ray connecting p to p_j. On this cone, whose (lateral) surface consists of geodesics emanating from p, there exists a geodesic lying tangent to the sphere $\partial\mathbf{b}(p_j, r)$. In other words, we take the cone of largest aperture 2α for which all geodesics pass through $\mathbf{b}(p_j, r)$.

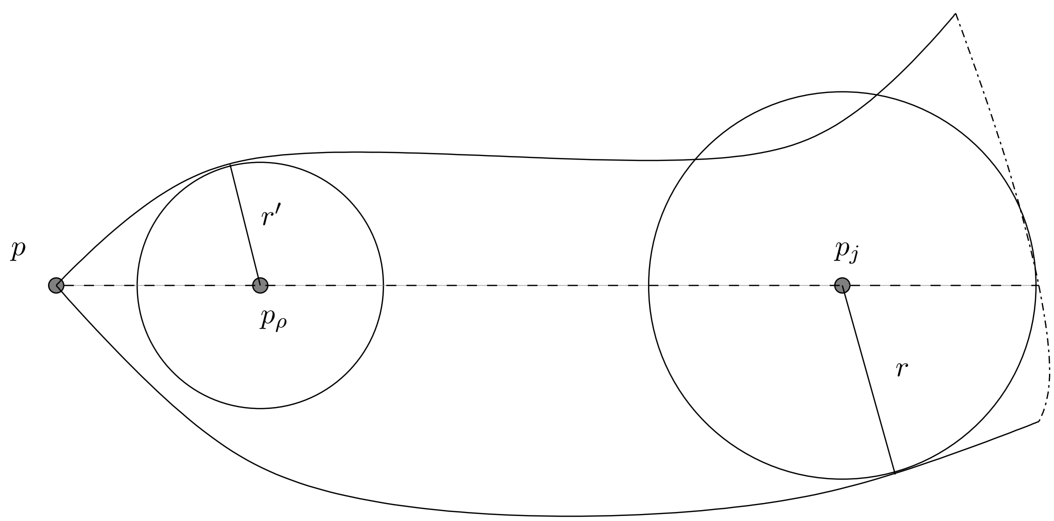

Figure 2: The largest cone with vertex p and with central axis the geodesic that connects p to p_j for which no geodesic lies outside of the ball $\mathbf{b}(p_j, r)$. The radius r' of the ball centered at p_ρ lying tangent to the cone is greater than $\frac{\Gamma_1 r \rho}{\Gamma_2 \rho_j}$

From this two things follow. First, by Lemma A.7, the distance from p to p_j, $\rho_j = \mathrm{dist}(p, p_j)$, the angle α, and the radius r are related by $r \leq \Gamma_2 \rho_j \sin\bigl(\min(\alpha, \pi/2)\bigr)$. Second, for $\rho \leq \rho_j$ and for a point p_ρ lying a distance of ρ from p along the central axis, the distance r' of p_ρ to the surface of the cone satisfies:

$$r' \geq \Gamma_1 \rho \sin\bigl(\min(\alpha, \pi/2)\bigr) \geq \frac{\Gamma_1 r \rho}{\Gamma_2 \rho_j}.$$

It follows the cone contains the ball $\mathbf{b}(p_\rho,(\Gamma_2\rho_j)^{-1}\Gamma_1 r\rho)$, which obviously is also contained in the convex hull of $\{p\}\cup\mathbf{b}(p_j,r)$.

Shifting to normal coordinates centered at p_j, rather than at p, we see that the geodesic ball $\mathbf{b}(p_\rho,(\Gamma_2\rho_j)^{-1}\Gamma_1 r\rho)$ contains the (image of the) Euclidean ball $B(\mathrm{Exp}_{p_j}^{-1}(p_\rho),(\Gamma_2^2\rho_j)^{-1}\Gamma_1 r\rho)$. Indeed, if $|y-\mathrm{Exp}_{p_j}^{-1}(p_\rho)|\le(\Gamma_2^2\rho_j)^{-1}\Gamma_1 r\rho$ then

$$\mathrm{dist}(\mathrm{Exp}_{p_j}(y),p_\rho)\le\Gamma_2|y-\mathrm{Exp}_{p_j}^{-1}(p_\rho)|\le\frac{\Gamma_1 r\rho}{\Gamma_2\rho_j}.$$

A straightforward argument using Euclidean geometry implies that the Euclidean convex hull of $\{\mathrm{Exp}_{p_j}^{-1}(p)\}\cup B(0,\Gamma_1 r/\Gamma_2^2)$ is contained in D_j. Hence, in the Euclidean metric, D_j is star shaped with respect to the ball $B(\mathrm{Exp}_{p_j}^{-1}(p_j),\Gamma_1 r/\Gamma_2^2)$. Moreover, since $\mathcal{D}_j\subset\mathbf{b}(p_j,R)$, we have that $D_j\subset B(\mathrm{Exp}_{p_j}^{-1}(p_j),R)$. Finally, from these facts it is easy to see that the bounds in (A.10) hold.

□

Applying this together with Proposition A.2 and Lemma 2.2 , we have the following result.

Proposition A.10. *Let $\mathcal{D}_j$ be as above, $m\in\mathbb{N}$ and $\mathsf{p}\in\mathbb{R}$, $1\le\mathsf{p}\le\infty$. Assume $m>d/\mathsf{p}$ when $\mathsf{p}>1$, and $m\ge d$, for $\mathsf{p}=1$. If $u\in W_\mathsf{p}^m(\mathcal{D}_j)$ satisfies $u|_X=0$, where X is a finite subset of $\mathcal{D}_j$, and if the geodesic meshnorm $h=h_X\le\frac{\Gamma_1^5F(\varphi)^2}{8m^2\Gamma_2^4}R$, then*

$$\|u\|_{W_\mathsf{p}^k(\mathcal{D}_j)}\le C_{m,k,\mathsf{p},\mathbb{M}}R^{m-k}F(\varphi)^{-d-2k}\|u\|_{W_\mathsf{p}^m(\mathcal{D}_j)} \tag{A.11}$$

$$\|u\|_{L_\infty(\mathcal{D}_j)}\le C_{m,k,\mathsf{p},\mathbb{M}}R^{m-d/\mathsf{p}}F(\varphi)^{-d}\|u\|_{W_\mathsf{p}^m(\mathcal{D}_j)}. \tag{A.12}$$

Proof. We first apply Proposition A.2 with $\tilde h$ the Euclidean meshnorm for $\mathrm{Exp}_{p_j}^{-1}X$, satisfying $\tilde h\le h/\Gamma_1\le\frac{d_{D_j}}{16m^2\gamma_{D_j}^2}\le\frac{2\Gamma_1^4RF(\varphi)^2}{16m^2\Gamma_2^4}$. Then, equations (A.1) and (A.2) hold for $u\circ\mathrm{Exp}_{p_j}$ on D_j, with γ_{D_j} and d_{D_j} replaced by the bounds in (A.10). Applying Lemma 2.2 then gives the bounds above. □

Theorem A.11 (Manifold Case). *Suppose that $\Omega\subseteq\mathbb{M}$ is a bounded, Lipschitz domain that satisfies a uniform cone condition, with the cone having radius $R_\Omega<\mathrm{r}_\mathbb{M}/3$ and angle φ. Let k, m, and p be as in Proposition A.2, and let $X\subset\Omega$ be a discrete set with mesh norm h satisfying*

$$h\le h_0R_\Omega,\quad h_0:=\frac{\Gamma_1^5}{8m^2\Gamma_2^4}F(\varphi)^2, \tag{A.13}$$

where Γ_1,Γ_2 and $F(\cdot)$ are defined in (2.3) and (A.3), respectively. If $u\in W_\mathsf{p}^m(\Omega)$ satisfies $u|_X=0$, then we have

$$\|u\|_{W_\mathsf{p}^k(\Omega)}\le C_{m,k,\mathsf{p},\mathbb{M}}F(\varphi)^{-(1+1/\mathsf{p})d-2m}h^{m-k}\|u\|_{W_\mathsf{p}^m(\Omega)} \tag{A.14}$$

and

$$\|u\|_{L_\infty(\Omega)}\le C_{m,\mathsf{p},\mathbb{M}}h^{m-d/\mathsf{p}}F(\varphi)^{-d+2d/\mathsf{p}-2m}\|u\|_{W_\mathsf{p}^m(\Omega)}. \tag{A.15}$$

Proof. We are given h in (A.13) to begin with. Thus, we may choose $R = \frac{8m^2\Gamma_2^4}{\Gamma_1^5} hF(\varphi)^{-2} \le R_\Omega$, and also take the $\mathcal{D}_j$'s to be the domains corresponding to this R. It follows that the conditions on h in Proposition A.10 hold; consequently,

$$\|u\|_{W_\mathsf{p}^k(\mathcal{D}_j)} \le C_{m,k,\mathsf{p},\mathbb{M}} h^{m-k} F(\varphi)^{-d-2m} \|u\|_{W_\mathsf{p}^m(\mathcal{D}_j)}, \tag{A.16}$$

$$\|u\|_{L_\infty(\mathcal{D}_j)} \le C_{m,\mathsf{p},\mathbb{M}} h^{m-d/\mathsf{p}} F(\varphi)^{-d+2d/\mathsf{p}-2m} \|u\|_{W_\mathsf{p}^m(\mathcal{D}_j)}. \tag{A.17}$$

Because of the decomposition $\Omega = \cup_{p_j \in T_r} \mathcal{D}_j$, the bound in (A.17) immediately implies (A.15). Moreover, this decomposition also gives us

$$\|u\|^\mathsf{p}_{W_\mathsf{p}^k(\Omega)} \le \sum_j \|u\|^\mathsf{p}_{W_\mathsf{p}^k(\mathcal{D}_j)} \le (C_{m,k,\mathsf{p},\mathbb{M}} h^{m-k} F(\varphi)^{-d-2m})^\mathsf{p} (\sum_j \|u\|^\mathsf{p}_{W_\mathsf{p}^m(\mathcal{D}_j)}).$$

From Definition 2.1, we see that

$$\sum_j \|u\|^\mathsf{p}_{W_\mathsf{p}^m(\mathcal{D}_j)} = \sum_{i=0}^m \int_\Omega \sum_j \chi_{\mathcal{D}_j}(p) |\nabla^i f|^\mathsf{p}_{g,p} \, \mathrm{d}\mu(p) \le \sup_{p\in\Omega} \left(\sum_j \chi_{\mathcal{D}_j}(p) \right) \|u\|^p_{W_\mathsf{p}^m(\Omega)}.$$

The sum $\sum_j \chi_{\mathcal{D}_j}(p)$ is precisely the number of $\mathcal{D}_j$'s that contain p. Since $\mathcal{D}_j \subset \mathbf{b}(p_j, R)$, p_j is itself also in $\mathbf{b}(p, R)$. Consequently, the number of $\mathcal{D}_j$'s containing p is bounded above by the number of balls $\mathbf{b}(p_j, r)$, where $r = 2RF(\varphi) \sim h/F(\varphi)$, that contain p, and, ultimately by the maximum number of $\mathbf{b}(p_j, r)$'s that can intersect each $\mathbf{b}(p, R)$. By Lemma A.5, this is $(4\alpha+1)^d e^{\frac{3(d-1)}{\sqrt{|\kappa|}} d_\mathbb{M}}$, where $\alpha = R/r = \frac{1}{2}F(\varphi)^{-1}$. Putting together the two previous inequalities then yields

$$\|u\|^\mathsf{p}_{W_\mathsf{p}^k(\Omega)} \le (C_{m,k,\mathsf{p},\mathbb{M}} h^{m-k} F(\varphi)^{-d-2m})^\mathsf{p} \, 2^{2d} F(\varphi)^{-d} e^{\frac{3(d-1)}{\sqrt{|\kappa|}} d_\mathbb{M}} \|u\|^\mathsf{p}_{W_\mathsf{p}^m(\Omega)}.$$

Taking the p^{th} root, lumping constants, and manipulating the result, we obtain (A.14). ☐

We remark that the various constants appearing in Theorem A.11, including h_0, only depend on φ, and only the right side of (A.13) depends on the radius R_Ω, and that dependence is linear. Thus, the dependence on Ω is completely explicit.

At this point we can extend the Duchon type error estimates for approximation by conditionally positive definite kernels. To our knowledge, this is the first result of this kind on bounded regions in compact Riemannian manifolds.

To this end, suppose $K : \Omega \times \Omega \to \mathbb{R}$ is positive definite (i.e. the matrix $\big(K(\xi,\zeta)\big)_{\xi,\zeta}$ is positive definite for each Ξ – see Definition 3.1) and consider that the "native space" $\mathcal{N}_K$, the reproducing kernel Hilbert space constructed by taking the space of arbitrary linear combinations of $K(\cdot,\xi)$, completed under the inner product $\langle f, g\rangle \mapsto \sum_{\xi,\zeta} A_\xi B_\zeta K(\xi,\zeta)$ for $f = \sum A_\xi K(\cdot,\xi)$ and $g = \sum B_\zeta K(\cdot,\zeta)$. In this case, it is well known that the kernel interpolant $I_\Xi f$ is the optimal interpolant in the sense of $\mathcal{N}_K$. Namely, $\|I_\Xi f\|_{\mathcal{N}_K} \le \|s\|_{\mathcal{N}_K}$ for all $s \in \mathcal{N}_K$ with $s|_\Xi = f|_\Xi$.

Corollary A.12. *Let $m > d/2$, and let K be a positive definite kernel on $\mathbb{M}$ for which $\mathcal{N}_K$ is continuously embedded in $W_2^m(\mathbb{M})$. Let $\Omega \subset \mathbb{M}$ satisfy a uniform cone condition with radius $R_\Omega \le \mathrm{r}_\mathbb{M}/3$ and angle φ. For $\Xi \subset \Omega$ having meshnorm $h \le h_0 R_\Omega$ and for $f \in \mathcal{N}_K$,*

$$\|f - I_\Xi f\|_{L_\infty(\Omega)} \le C_K F(\varphi)^{-2m} h^{m-d/2} \|f\|_{\mathcal{N}_K(\mathbb{M})}.$$

We note that this result holds for a much larger class of kernels than considered in the previous sections (i.e., defined by Definition 3.2). In particular, there are numerous examples of compactly supported kernels on $\mathbb{R}^d$ and $\mathbb{S}^d$ having native spaces that are Sobolev spaces, but which do not invert an elliptic differential operator. However, his type of error estimate should be compared to those in Corollary 5.9 and Corollary 5.10 – observe that the condition on the target function is quite restrictive (it needs to be in $\mathcal{N}_K$ and there is a basic disagreement between the approximation order $m - d/2$ and the smoothness assumption).

Proof. Apply Theorem A.11 to $u = f - I_\Xi$, we see that

$$\begin{aligned}
\|f - I_\Xi f\|^2_{L_\infty(\Omega)} &\le \left(C_{m,\mathbb{M}} h^{m-d/\mathsf{p}} F(\varphi)^{-2m}\right)^2 \|f - I_\Xi f\|^2_{W_2^m(\Omega)} \\
&\le \left(C_{m,\mathbb{M}} h^{m-d/\mathsf{p}} F(\varphi)^{-2m}\right)^2 \|f - I_\Xi f\|^2_{W_2^m(\mathbb{M})} \\
&\le C\left(C_{m,\mathbb{M}} h^{m-d/\mathsf{p}} F(\varphi)^{-2m}\right)^2 \|f - I_\Xi f\|^2_{\mathcal{N}_K} \\
&\le C_K F(\varphi)^{-4m} h^{2m-d} \|f\|^2_{\mathcal{N}_K}.
\end{aligned}$$

The next to last inequality is the embedding $\mathcal{N}_K \subset W_2^m(\mathbb{M})$, while the last inequality is the Pythagorean theorem for orthogonal projectors $\|f - I_\Xi f\|^2_{\mathcal{N}_K} + \|I_\Xi f\|^2_{\mathcal{N}_K} = \|f\|^2_{\mathcal{N}_K}$. □

There are several domains that are important for us, and that we will discuss below. We begin with the manifold itself. In that case, we may take $R_\Omega = \mathrm{r}_\mathbb{M}/3$. The angle φ may be set equal to $\pi/2$, because every such cone is contained in $\mathbb{M}$. This means that $F(\varphi) = F(\pi/2) = 1/8$.

Corollary A.13 (Full Manifold). *Suppose that $\mathbb{M}$ is compact. Let k, m, and p be as in Theorem A.11. Then, with $h_0 := \frac{\Gamma_2^2}{8^3 m^2 \Gamma_1^2}$, there is a constant $C_{m,k,d,\mathbb{M}}$ such that if $X \subset \mathbb{M}$ has mesh norm $h \le h_0 \mathrm{r}_\mathbb{M}/3$ and if $u \in W_\mathsf{p}^m(\mathbb{M})$ satisfies $u|_X = 0$, then*

$$\|u\|_{W_\mathsf{p}^k(\mathbb{M})} \le C_{m,k,\mathsf{p},\mathbb{M}} h^{m-k} \|u\|_{W_\mathsf{p}^m(\mathbb{M})}. \tag{A.18}$$

The domains that we now turn to are balls, annuli, and complements of balls. In all of the cases discussed below, the domains Ω satisfy the ball property described in Corollary A.8 at each point $q \in \partial\Omega$. Consequently, we may take $\varphi = \arcsin(\frac{1}{2\Gamma_2})$, and so $F(\varphi) = 1/(8\Gamma_2 + 4)$. It thus follows that in all such cases

$$h_0 = \frac{\Gamma_1^5}{128 m^2 \Gamma_2^4 (2\Gamma_2 + 1)^2}. \tag{A.19}$$

and all of the factors in Theorem A.11 depend only on parameters from the manifold itself, as well as p, k, m, but not at all on the center and the radius of the ball/annulus/punctured ball.

We now turn to balls. A ball of any size may be treated; however, if the the radius is larger that $\mathrm{r}_{\mathbb{M}}$ it may intersect itself, giving rise to corners with angles that have to be dealt with on a case by case basis. This phenomenon is easy to see in the case of the torus embedded in $\mathbb{R}^3$, where a sufficiently large ball begins to wrap back on itself. When the radius of the ball is less than $\mathrm{r}_{\mathbb{M}}$, this wrapping doesn't happen. With this assumption, we have the following result:

Corollary A.14 (Zeros estimate on balls). *Assume $m > d/2$. Suppose that $r < \mathrm{r}_{\mathbb{M}}$. If $u \in W_p^m(\mathbb{M})$ vanishes on $X \subset \mathbf{b}(p, r)$, where $h \leq h_0 r/2$, where h_0 is given by (A.19), we have*

$$\|u\|_{W_p^k(\mathbf{b}(p,r))} \leq C_{m,k,\mathrm{p},\mathbb{M}} h^{m-k} |u|_{W_p^m(\mathbf{b}(p,r))}.$$

Proof. Obviously every point $q \in \partial \mathbf{b}(p, r)$ satisfies the conditions in Corollary A.8. A direct application of Theorem A.11 then completes the proof. □

Corollary A.15 (Hölder estimate on balls). *If m is greater than $d/2 + \epsilon$, and the conditions of Corollary A.14hold (in particular, r is less than $\mathrm{r}_{\mathbb{M}}$, and $h \leq h_0 r$), and if $u \in W_2^m(\mathbf{b}(p, r))$ satisfies $u|_X = 0$, then for every $z \in \mathbf{b}(p, r)$,*

$$|u(p) - u(z)| \leq C r^{m-\epsilon-d/2} \mathrm{dist}(p, z)^\epsilon \|u\|_{W_2^m(\mathbf{b}(p,r))},$$

where C is a constant depending only on m, $\mathbb{M}$ and ϵ.

Proof. This follows because the Sobolev embedding theorem, in conjunction with Lemma 2.2 ensures that $\tilde{w} = w \circ \mathrm{Exp}_p \in C^\epsilon(B(0, \mathrm{r}_{\mathbb{M}}))$. Thus for $z = \mathrm{Exp}_p(x) \in \mathbf{b}(p, \mathrm{r}_{\mathbb{M}})$,

$$\frac{|w(p) - w(z)|}{\mathrm{dist}(p, z)^\epsilon} = \frac{|\tilde{w}(0) - \tilde{w}(x)|}{|x|^\epsilon} \leq |\tilde{w}|_{C^\epsilon(B(0,\mathrm{r}_{\mathbb{M}}))} \leq C \|\tilde{w}\|_{W_2^m(B(0,\mathrm{r}_{\mathbb{M}}))}.$$

For a general $r < \mathrm{r}_{\mathbb{M}}$, set $\tilde{w}(\frac{\mathrm{r}_{\mathbb{M}}}{r} x) = \tilde{u}(x)$. Then

$$\frac{|u(p) - u(z)|}{\mathrm{dist}(p, z)^\epsilon} \leq \left(\frac{\mathrm{r}_{\mathbb{M}}}{r}\right)^\epsilon C \|\tilde{w}\|_{W_2^m(B(0,\mathrm{r}_{\mathbb{M}}))} = \left(\frac{\mathrm{r}_{\mathbb{M}}}{r}\right)^\epsilon C \left(\sum_{k \leq m} \left(\frac{r}{\mathrm{r}_{\mathbb{M}}}\right)^{2k} |\tilde{u}|^2_{W_2^k(B(0,r))} \right)^{1/2}.$$

The result follows by applying Lemma 2.2 in conjunction with Corollary A.14. □

A similar argument to the proof of Corollary A.14 given above yields these results for annuli and complements of balls. In the following two lemmas, we are concerned with the case $p = 2$. Consequently, we suppress dependence on these parameters by expressing the constant from the zeros lemma for such domains simply as Λ. In other words,

$$\Lambda := \max_{k=0\ldots m-1} C_{m,k,2,\mathbb{M}} \left(8\Gamma_2 + 4\right)^{(3/2)d+2m}, \tag{A.20}$$

which depends only on m and $\mathbb{M}$.

Corollary A.16 (Zeros lemma on annuli). *Assume $m > d/2$, Let $\mathbf{a} = \mathbf{b}(p,r) \setminus \mathbf{b}(p, r-t)$, where $0 < t < r < \mathrm{r}_{\mathbb{M}}$, and let h_0 be given by (A.19). If $u \in W_p^m(\mathbf{a})$ vanishes on $X \subset \mathbf{a}$, where $h \le h_0 \min(t/2, \mathrm{r}_{\mathbb{M}}/3)$, we have*

$$\|u\|_{W_2^k(\mathbf{a})} \le \Lambda h^{m-k} |u|_{W_p^m(\mathbf{a})}.$$

Proof. At each point q of the boundary of $\mathbf{a}$ an open ball of radius $t/2$ can be placed inside $\mathbf{a}$ with a boundary that passes through q. The result follows from Corollary A.8 and Theorem A.11. □

Corollary A.17 (Zeros lemma on complements of balls). *If $r < \mathrm{r}_{\mathbb{M}}/3$ and if $u \in W_2^m(\mathbb{M})$ vanishes on X with $h = h(X, \mathbf{b}(p,r)^c) \le h_0 \mathrm{r}_{\mathbb{M}}/3$, then*

$$\|u\|_{W_2^{m-k}(\mathbf{b}(p,r)^c)} \le \Lambda h^{m-k} |u|_{W_2^m(\mathbf{b}(p,r)^c)}.$$

Proof. By placing its center, q, a distance of $r + 2\mathrm{r}_{\mathbb{M}}/3$ away from p, the ball $\mathbf{b}(q, 2\mathrm{r}_{\mathbb{M}}/3)$ can be placed in $\mathbf{b}(p,r)^c$. It follows that the set satisfies a cone condition with radius $R = \mathrm{r}_{\mathbb{M}}/3$. □

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